$$-b + \sqrt{b^2 - \cos^2 a}$$

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$$(x + y)$$

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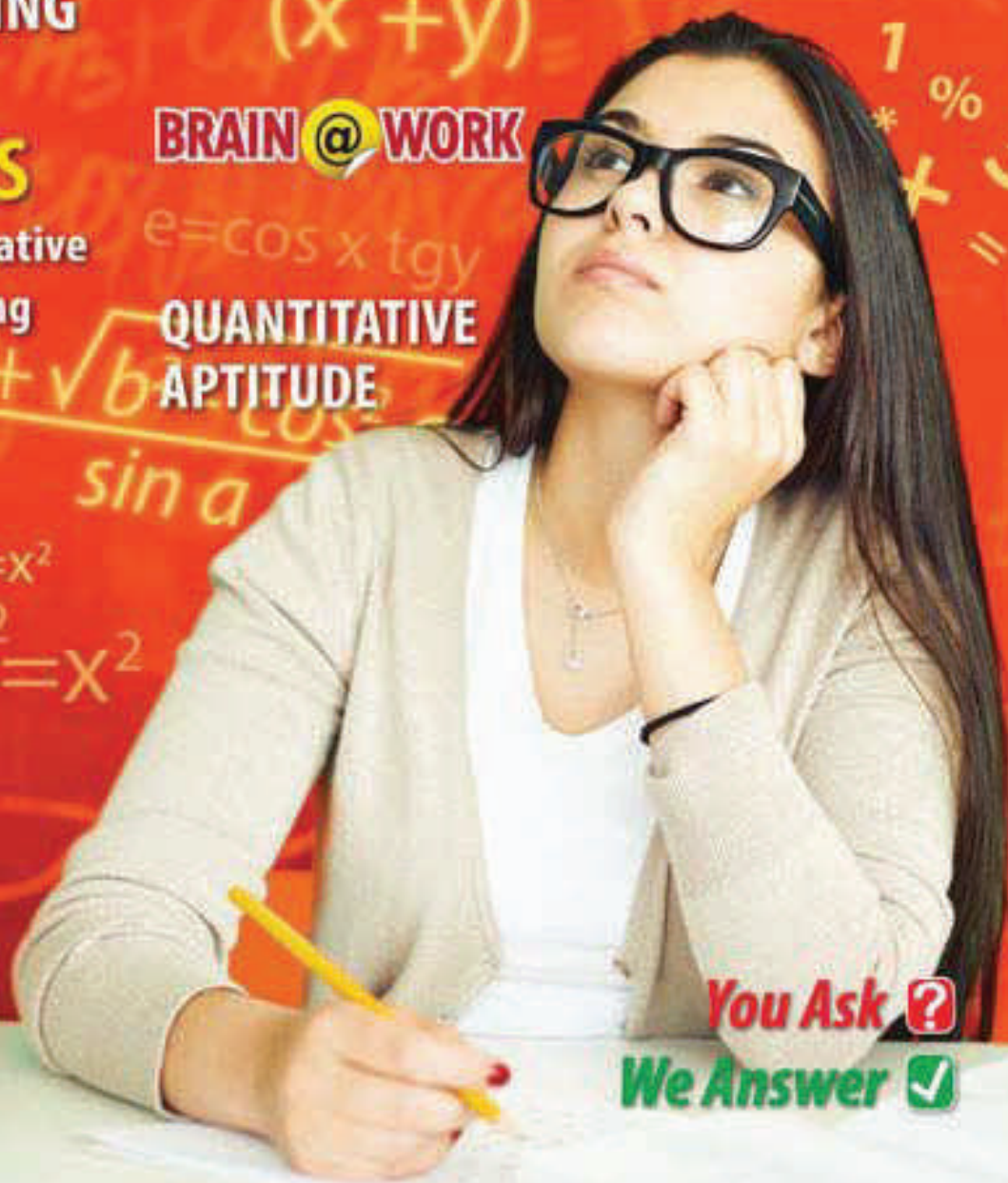
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TO OUR READERS

We are happy that intelligent students, teachers and other professionals continue to patronise Mathematics Today, Chemistry Today, Physics For You and Biology Today.

To them, we are addressing this open letter in view of increase in the cost of production and postage in the last five years. All round spiralling prices have pushed production costs so high, that many in out fraternity find it impossible to continue business. We are compelled to raise the price to ₹ 40 from July 2016 issue.

We understand the pressure of cost on the student-teacher community in general but, we are hoping our readers will understand our problems and that we have no option but to comply with this unavoidable move.

We on our part, will keep up our efforts to improve the magazines in all its aspects.

MATHS MUSING

PROBLEM Set 162

JEE MAIN

1. If the roots of the equation $ax^2 + 2bx + c = 0$ are real and distinct, then the roots of $(a + c)(ax^2 + 2bx + c) = 2(ac - b^2)(x^2 + 1)$ are
(a) real and distinct
(b) real and equal
(c) imaginary and distinct
(d) imaginary and equal

2. If x_1, x_2, x_3, x_4 are the roots of the equation $x^4 - x^3 \sin 2\alpha + x^2 \cos 2\alpha - x \cos \alpha - \sin \alpha = 0$, $\alpha \neq \dfrac{\pi}{6}$, then $\displaystyle\sum_{i=1}^{4} \tan^{-1} x_i =$
(a) α
(b) $\dfrac{\pi}{2} - \alpha$
(c) $-\alpha$
(d) $\pi - \alpha$

3. The number of non-zero values of z such that $\bar{z} = z^4$ is
(a) 5 (b) 6 (c) 8 (d) 9

4. A fair coin is tossed 13 times. The probability of getting atleast 7 consecutive heads is
(a) $\dfrac{3}{16}$ (b) $\dfrac{3}{32}$ (c) $\dfrac{1}{16}$ (d) $\dfrac{1}{32}$

5. In a triangle ABC, $C = \dfrac{\pi}{2}$ and D is a point on the side CB. The circle with centre D and radius 1 touches the sides AB and AC. If $s = 4$, then Δ is
(a) $\dfrac{4}{3}$
(b) 2
(c) $\dfrac{8}{3}$
(d) $\dfrac{10}{3}$

JEE ADVANCED

6. Let a and b be natural numbers such that $a < b$ and H.M. of a and b is 2014 then $a =$

(a) 1368
(b) 1060
(c) 1026
(d) 1008

COMPREHENSION

P is a point on the ellipse $x^2 + 2y^2 = 2$ with foci S and S_1.

7. The locus of the orthocentre of triangle PSS_1 is $(2 - x^2)y^2 = \lambda(1 - x^2)^2$, where $\lambda =$
(a) 1
(b) 2
(c) 3
(d) 4

8. The locus of the incentre of triangle PSS_1 is a conic of latus rectum
(a) $3 + 2\sqrt{2}$
(b) $5 - 2\sqrt{2}$
(c) $6 - 4\sqrt{2}$
(d) $4 + 2\sqrt{2}$

INTEGER MATCH

9. If $A_k = \dfrac{k(k-1)}{2} \cos \dfrac{k(k-1)}{2} \pi$ and $\displaystyle\sum_{k=1}^{101} A_k = S$, then the sum of the digits of S is

MATRIX MATCH

10. In a triangle ABC with integer sides, $C = \dfrac{\pi}{2}$. The values of the side a are given in column I and the numbers of such triangles are in column II.

Column I		Column II	
(P)	9	(1)	1
(Q)	10	(2)	2
(R)	12	(3)	3
(S)	20	(4)	4

	P	Q	R	S		P	Q	R	S
(a)	2	1	4	4	(b)	2	1	4	3
(c)	1	2	3	4	(d)	3	4	1	2

BRAIN @ WORK

Topic : Quadratic Equations and Expressions

This article is a collection of shortcut methods, important formulas and MCQ's along with their detailed solutions which provides an extra edge to the readers who are preparing for various competitive exams like JEE(Main & Advanced) and other PET's.

IDENTITY

An identity is the statement of equality between two expressions which is always true for all values of the variables involved. So, $f(x) = g(x)$ is an identity, if $f(x)$ and $g(x)$ have same value for every value of x.

Note:

(i) A polynomial of degree n represents an identity, if it is satisfied by $(n + 1)$ or more values of x.

(ii) If $f(x) = g(x)$ represents an identity, then the coefficients of similar terms of x are equal.

(iii) If an equation $ax^3 + bx^2 + cx + d = 0$ represents an identity in terms of x, then $a = b = c = d = 0$.

ROOTS OF EQUATION

(i) Factor theorem

    (a) I. $(x - \alpha)$ is a factor of a polynomial $f(x)$ if and only if $f(\alpha) = 0$.

        II. $(x - \alpha)$ $(x - \beta)$ is a factor of a polynomial $f(x)$ iff $f(\alpha) = 0$ as well as $f(\beta) = 0$.

    (b) $(x - \alpha)^2$ is a factor of a polynomial $f(x)$ if and only if $f(\alpha) = f'(\alpha) = 0$. In this case, we say that α is a repeated root of $f(x) = 0$ (a double root). Physically this means that the x-axis is tangent to the curve $y = f(x)$ at $x = \alpha$.

    (c) If $(x - \alpha)^m$ is factor of a polynomial $f(x) = 0$ then $f(\alpha) = f'(\alpha) = f''(\alpha) = f'''(\alpha) = ... = f^{m-1}(\alpha) = 0$ and $f^m(\alpha) \neq 0$

(ii) Remainder theorem : If a polynomial $f(x)$ is divided by $(x - \alpha)$, then the value of the remainder is $f(\alpha)$. Also, it is easy to show that the value of remainder, when $f(x)$ is divided by $(x - \alpha)$ $(x - \beta)$, is

$$\left(\frac{f(\alpha) - f(\beta)}{\alpha - \beta} \right)x + \left(\frac{\alpha f(\beta) - \beta f(\alpha)}{\alpha - \beta} \right)$$

(iii) Position of roots of a polynomial equation :

    If $f(x) = 0$ is an equation and a, b are two real numbers such that

(a) $f(a) f(b) < 0$, then the equation $f(x) = 0$ has at least one real root or an odd number of real roots between a and b.

(b) If $f(a)$ and $f(b)$ are of the same signs, then either no real root or an even number of real roots of $f(x) = 0$ lie between a and b.

RELATION BETWEEN ROOTS AND COEFFICIENTS OF POLYNOMIAL EQUATION

Consider the general equation of nth degree
$f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n = 0$, where $a_0, a_1, ..., a_n \in R$ & $n \in W$
Let its roots be $\alpha_1, \alpha_2, ..., \alpha_n$. Then

(i) Sum of roots taken one at a time $= S_1 = \sum \alpha_i = -\dfrac{a_1}{a_0}$

    Sum of product of roots taken two at a time

$$= S_2 = \sum_{i \neq j} \alpha_i \alpha_j = +\frac{a_2}{a_0}$$

    Sum of product of roots taken three at a time

$$= S_3 = \sum_{i \neq j \neq k} \alpha_i \alpha_j \alpha_k = -\frac{a_3}{a_0}$$

    ....
    ....

    The product of roots taken all at a time

$$= S_n = \alpha_1 \alpha_2 ... \alpha_n = (-1)^n \cdot \frac{a_n}{a_0}$$

    Number of terms in $S_1, S_2, S_3, ..., S_n$ are respectively $^nC_1, {}^nC_2, {}^nC_3, ..., {}^nC_n$.

(ii) $f(\alpha_1) = f(\alpha_2) = f(\alpha_3) = f(\alpha_4) = ... = f(\alpha_n) = 0$

(iii) $a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n = a_0(x - \alpha_1)$ $(x - \alpha_2)$ $(x - \alpha_3) ..... (x - \alpha_{n-1})$ $(x - \alpha_n)$

(iv) If $f(x) = 0$ has n real roots, then $f'(x) = 0$ has $(n - 1)$ real roots.

(v) If $f(x) = 0$ has n real roots, then $f(x) = a_0(x - \alpha_1)$ $(x - \alpha_2)$ $(x - \alpha_3) ..... (x - \alpha_{n-1})$ $(x - \alpha_n)$

(vi) If $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are n roots of an equation, then the equation can be written as
$x^n - S_1 x^{n-1} + S_2 x^{n-2} + ... + (-1)^n S_n = 0$.
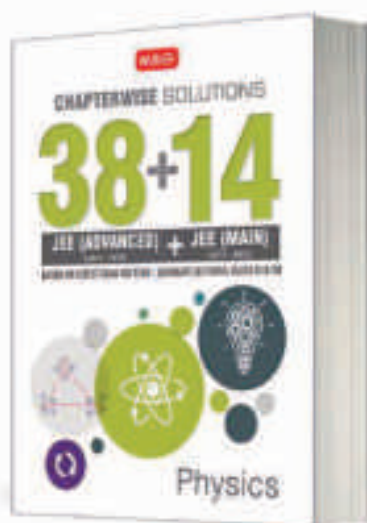
Sanjay Singh Mathematics Classes, Chandigarh

TRANSFORMATION OF EQUATIONS

(i) An equation whose roots are reciprocals of the roots of a given equation is obtained by replacing x by $1/x$ in the given equation and simplify it to make it a polynomial equation.

(ii) An equation whose roots are negative of the roots of a given equation is obtained by replacing x by $-x$ in the given equation and simplify it to make it a polynomial equation.

(iii) An equation whose roots are squares of the roots of a given equation is obtained by replacing x by $\sqrt{x}$ in the given equation and simplify it to make it a polynomial equation.

(iv) An equation whose roots are cubes of the roots of a given equation is obtained by replacing x by $x^{1/3}$ in the given equation and simplify it to make it a polynomial equation.

QUADRATIC EQUATION

- An equation that can be written in the form $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$, is called a quadratic equation.

- The solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

- The quantity $b^2 - 4ac$ is called the discriminant of the quadratic equation and is denoted by D or Δ.

- Usually, the two roots of $ax^2 + bx + c = 0$ are denoted by α and β. The expression $ax^2 + bx + c$ can thus be written as $ax^2 + bx + c = a(x - \alpha)(x - \beta)$.

Sum and Product of Roots

- Sum of roots $= S = \alpha + \beta = \dfrac{-b}{a} = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

- Product of roots $= P = \alpha\beta = \dfrac{c}{a} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

  Also, (Difference of roots)$^2 = (\alpha - \beta)^2$

$$= (\alpha + \beta)^2 - 4\alpha\beta = \frac{b^2 - 4ac}{a^2}$$
$$\Rightarrow \quad |\alpha - \beta| = \frac{\sqrt{D}}{|a|}$$

- The quadratic equation with sum of roots S and product of roots P is given by $x^2 - Sx + P = 0$, i.e. $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Nature of the Roots

(i) The roots are real and distinct iff $D > 0$.

(ii) The roots are real and equal iff $D = 0$ and the equal root is given by $x = -b/2a$.
    When $D = 0$, $ax^2 + bx + c$ is a perfect square, under this condition, we have
$$ax^2 + bx + c = \left\{ \sqrt{a}\left(x + \frac{b}{2a} \right) \right\}^2$$

(iii) The roots are complex with non-zero imaginary part iff $D < 0$.

(iv) The roots are rational iff a, b, c are rational and D is a perfect square.

(v) The roots are of the form $p + \sqrt{q}$ $(p, q \in Q)$ i.e. irrational iff a, b, c are rational and D is not a perfect square.

(vi) If a quadratic equation in x has more than two roots, then it is an identity in x i.e., $a = b = c = 0$.

Nature of the Roots of $P(x) \cdot Q(x) = 0$

If D_1 and D_2 are the discriminants of the quadratic equations $P(x) = 0$ and $Q(x) = 0$, then the following possibilities arise about the roots of the equation $P(x) \cdot Q(x) = 0$.

(i) If $D_1 + D_2 \geq 0$, then there will be at least two real roots of the equation $P(x) \cdot Q(x) = 0$.

(ii) If $D_1 + D_2 < 0$, then there will be at least two imaginary roots of $P(x) \cdot Q(x) = 0$.

(iii) If $D_1 \cdot D_2 < 0$, then the equation $P(x) \cdot Q(x) = 0$ will have two real roots.

(iv) If $D_1 \cdot D_2 > 0$, then the equation $P(x) \cdot Q(x) = 0$ has either four real roots or no real root.

Condition for Common Roots

S.No.	Type of Situation	Required Conditions
1.	$ax^2 + bx + c = 0$ and $a_1 x^2 + b_1 x + c_1 = 0$ have one common root.	$(ab_1 - a_1 b)(bc_1 - cb_1) = (a_1 c - ac_1)^2$
2.	$ax^2 + bx + c = 0$ and $a_1 x^2 + b_1 x + c_1 = 0$ have both roots in common.	$\dfrac{a}{a_1} = \dfrac{b}{b_1} = \dfrac{c}{c_1}$
3.	If the two equations $ax^2 + bx + c = 0$, $a_1 x^2 + b_1 x + c_1 = 0$ with real coefficients have an imaginary root common, then both roots will be common.	$\dfrac{a}{a_1} = \dfrac{b}{b_1} = \dfrac{c}{c_1}$
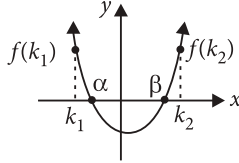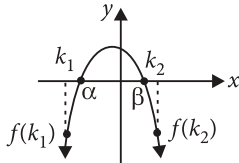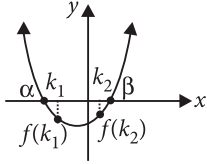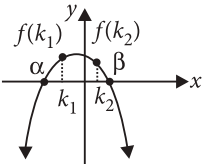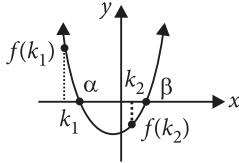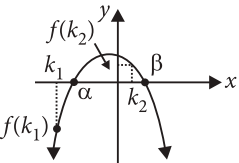
4.	If the two equations $ax^2 + bx + c = 0$; $a_1x^2 + b_1x + c_1 = 0$ with rational coefficients have an irrational root common, then both roots will be common.	$\dfrac{a}{a_1} = \dfrac{b}{b_1} = \dfrac{c}{c_1}$
5.	If every pair of three quadratic equations have a common root.	Roots are taken as α, β ; β, γ ; γ, α
6.	A quadratic equation and cubic equation have a common root.	Try to find the root of cubic equation by factorization.

POSITION OF ROOTS OF THE QUADRATIC EQUATION $ax^2 + bx + c = 0$

(i) With Respect to One Quantity (k)

S.No.	Situation	Graphical Representation	Required Conditions
1.	Both the roots are less than k i.e., $\alpha < \beta < k$		(i) $D \geq 0$ (ii) $af(k) > 0$ (iii) $k > \dfrac{-b}{2a}$
2.	Both the roots are greater than k i.e., $k < \alpha < \beta$		(i) $D \geq 0$ (ii) $af(k) > 0$ (iii) $k < \dfrac{-b}{2a}$
3.	k lies between the roots i.e. $\alpha < k < \beta$		(i) $D > 0$ (ii) $af(k) < 0$

(ii) With Respect to Two Quantities k_1 and k_2

S.No.	Situation	Graphical Representation	Required Conditions
1.	Distinct roots lie in the interval (k_1, k_2) i.e., $k_1 < \alpha < \beta < k_2$	$a > 0$; $a < 0$	(i) $D > 0$ (ii) $af(k_1) > 0$ (iii) $af(k_2) > 0$ (iv) $k_1 < \dfrac{-b}{2a} < k_2$
2.	Interval (k_1, k_2) lies between the roots i.e., $\alpha < k_1 < k_2 < \beta$	$a > 0$; $a < 0$	(i) $D > 0$ (ii) $af(k_1) < 0$ (iii) $af(k_2) < 0$
3.	One root lies in the interval (k_1, k_2) i.e., $k_1 < \alpha < k_2 < \beta$	$a > 0$; $a < 0$	(i) $D > 0$ (ii) $f(k_1)\,f(k_2) < 0$

PROBLEMS

1. If a, b and c are unequal positive real numbers such that $2b = a + c$, then the roots of $ax^2 + 2bx + c = 0$ are
   (a) real and equal (b) real and distinct
   (c) imaginary (d) none of these

2. The equation $x^2 + ax - a^2 - 1 = 0$ will have roots of opposite signs if
   (a) $a \in (-\infty, \infty)$
   (b) $a \in [-1, 1]$
   (c) $a \in (-\infty, -1) \cup (1, \infty)$
   (d) None of these

3. If a, b and c are odd integers and $ax^2 + bx + c = 0$ has real roots, then
   (a) both roots are rational
   (b) both roots are irrational
   (c) both roots are positive
   (d) roots are of opposite signs

4. If both the roots of $x^2 - ax + a = 0$ are greater than 2, then complete set of values of 'a' is
   (a) $a \in (-\infty, 4)$ (b) $a \in (0, 2)$
   (c) $a \in (4, \infty)$ (d) none of these

5. If both roots of $ax^2 + ax + 1 = 0$ are less than one then

   (a) $a \in \left(-\dfrac{1}{2}, 0\right] \cup [4, \infty)$
   (b) $a \in (-\infty, 0) \cup [4, \infty)$
   (c) $a \in \left(-\dfrac{1}{2}, 0\right] \cup (0, \infty)$
   (d) $\left(-\infty, -\dfrac{1}{2}\right] \cup [4, \infty)$

6. Total number of integral values of 'a' such that $x^2 + ax + a + 1 = 0$ has integral roots is equal to
   (a) one   (b) two
   (c) three   (d) four

7. If $x^2 + 2ax + a < 0 \ \forall \ x \in [1, 2]$ then
   (a) $a \in \left(-\infty, -\dfrac{4}{5}\right)$ (b) $a \in \left(-\dfrac{4}{5}, -\dfrac{1}{3}\right)$
   (c) $a \in \left(-\infty, -\dfrac{1}{3}\right)$ (d) none of these

8. If both the roots of $x^2 + ax + 2 = 0$ belong to the interval $(0, 3)$, then exhaustive range of 'a' is
   (a) $(-6, 0)$ (b) $\left(-\dfrac{11}{3}, -2\right]$
   (c) $\left[-\dfrac{11}{3}, 0\right)$ (d) none of these

9. If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have a common root and $a, b, c \in N$, then minimum value of $a + b + c$ is equal to

(a) 3   (b) 9

(c) 6   (d) 12

10. If two roots of $x^3 - ax^2 + bx - c = 0$ are equal in magnitude but opposite in signs, then

(a) $a + bc = 0$ (b) $a^2 = bc$

(c) $ab = c$ (d) $a - b + c = 0$

11. If $a, b, c \in R$ such that $a + b + c = 0$ and $a \neq c$, then the roots of $(b + c - a) x^2 + (c + a - b) x + (a + b - c) = 0$ are

(a) real and equal   (b) real and distinct

(c) imaginary   (d) None of these

12. If $\sin\theta$, $\cos\theta$ are the roots of $ax^2 + bx + c = 0$, then

(a) $a^2 = b^2 + 2ab$ (b) $b^2 = a^2 + 2ab$

(c) $a^2 = b^2 + 2ac$ (d) $b^2 = a^2 + 2ac$

13. Both roots of $(a^2 - 1) x^2 + 2ax + 1 = 0$ belong to the interval $(0, 1)$, then exhaustive set of values of 'a' is

(a) $(-\infty, -2) \cup (-1, \infty)$

(b) $(-\infty, -2)$

(c) $(-\infty, 0) \cup \left(0, \dfrac{-1 + \sqrt{5}}{2}\right)$

(d) None of these

14. The complete set of values of 'a' such that $x^2 + ax + a^2 + 6a < 0 \ \forall \ x \in [-1, 1]$ is

(a) $\left(\dfrac{-5 - \sqrt{21}}{2}, \dfrac{-7 + \sqrt{45}}{2}\right)$

(b) $\left(\dfrac{-7 - \sqrt{45}}{2}, \dfrac{-5 - \sqrt{21}}{2}\right)$

(c) $\left(\dfrac{-5 + \sqrt{21}}{2}, \dfrac{-7 + \sqrt{45}}{2}\right)$

(d) None of these

15. If $t - 1$ and $-t - 1$, for $t \in R$ are the roots of $(a + 2) x^2 + 2ax - 1 = 0$, then complete set of values of 'a' is

(a) ϕ (b) $(-\infty, \infty)$

(c) $(-\infty, 0)$ (d) $(0, \infty)$

16. If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root and a, b, c are non zero real numbers, then $\dfrac{a^3 + b^3 + c^3}{abc}$ is equal to

(a) 1   (b) 2

(c) 3   (d) None of these

17. If roots of the equation $ax^3 + bx^2 + cx + d = 0$ remain unchanged by increasing each coefficient by one unit, then

(a) $a = b = c = d \neq 0$ (b) $a = b \neq c = d \neq 0$

(c) $a \neq b = c = d \neq 0$ (d) $a = b = c \neq d \neq 0$

18. $\tan\theta$ and $\sec\theta$ are the roots of equation $ax^2 + bx + c = 0$, then

(a) $a^4 = b^2 (-2ac + b^2)$

(b) $b^4 = a^2 (2ac + a^2)$

(c) $a^4 = b^2 (-4ac + b^2)$

(d) $b^4 = a^2 (4ac + a^2)$

19. If $x^3 + ax + 1 = 0$ and $x^4 + ax^2 + 1 = 0$ have a common root, then complete set of values of a is

(a) $(-\infty, -2)$ (b) $\{-2\}$

(c) $[-2, \infty)$ (d) None of these

20. a, b, c are distinct positive real numbers such that $b(a + c) = 2ac$, then the roots of $ax^2 + 2bx + c = 0$ are

(a) real and equal   (b) real and distinct

(c) imaginary   (d) None of these

21. Complete set of values of 'a' such that $\dfrac{x^2 - x}{1 - ax}$ attains all real values is

(a) $[1, 4]$ (b) $(0, 4]$

(c) $(0, 1]$ (d) $[1, \infty)$

22. The equations $ax^2 - 2bx + c = 0$, $bx^2 - 2cx + a = 0$ and $cx^2 - 2ax + b = 0$ will have only positive roots, provided

(a) $a = b = c$ (b) $a \neq b \neq c$

(c) $a \neq b = c$ (d) $a = b \neq c$

23. If $a \in R^-$ and $a \neq -2$ then the equation $x^2 + a |x| + 1 = 0$,

(a) can not have any real root

(b) must have exactly two real roots

(c) must have either exactly two real roots or no real roots

(d) must have either four real roots or no real roots.

24. If $x^2 + ax + 1$ is a factor of $ax^3 + bx + c$ then
(a) $b + a + a^3 = 0$, $a^2 + c = 0$
(b) $b - a + a^3 = 0$, $a^2 + c = 0$
(c) $b + a - a^3 = 0$, $a^2 + c = 0$
(d) None of these

25. If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals then $\dfrac{b^2}{ac} + \dfrac{bc}{a^2}$ is equal to

(a) 1   (b) –1   (c) 2   (d) –2

26. The least value of 'n' such that $(n - 2) x^2 + 8x + n + 4 > 0 \ \forall \ x \in R$, where $n \in N$, is
(a) 3   (b) 5
(c) 4   (d) None of these

27. If both the roots of $x^2 - 2ax + a^2 + a - 3 = 0$ are less than three then
(a) $a \in (-\infty, 2)$ (b) $a \in (4, \infty)$
(c) $a \in [2, 3]$ (d) $a \in (3, 4]$

28. If the roots of $(x - a) (x - b) = f$ are c and d then the roots of $(x - c) (x - d) + f = 0$ are

(a) a and b (b) $\dfrac{a}{f}, \dfrac{b}{f}$

(c) $\dfrac{f}{a}, \dfrac{f}{b}$ (d) $\dfrac{c}{f}, \dfrac{d}{f}$

29. If the roots of $x^2 - ax + b = 0$ differ by unity then
(a) $b^2 = 1 + 4a$ (b) $a^2 = 1 + 4b$
(c) $b^2 + 4a = 1$ (d) $a^2 + 4b = 1$

30. Let a, b, c are positive real numbers forming a G.P. If $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then $\dfrac{d}{a}, \dfrac{e}{b}, \dfrac{f}{c}$ are in
(a) G.P.   (b) A.P.
(c) H.P.   (d) None of these

31. The maximum value of the expression
$y = 2(a - x) (x + \sqrt{x^2 + b^2})$ is equal to
(a) $|a^2 - b^2|$ (b) $a^2 + 2b^2$
(c) $2a^2 + b^2$ (d) $a^2 + b^2$

32. If $x^2 + ax + bc = 0$ and $x^2 + bx + ca = 0$, $(a \neq b)$ have a common root, then their other roots satisfy the equation
(a) $x^2 + ax + ab = 0$ (b) $x^2 + bx + ab = 0$
(c) $x^2 + cx + ab = 0$ (d) $x^2 + abx + c = 0$

33. If $x^2 - (a - 3) x + a = 0$ has at least one positive root then
(a) $a \in (-\infty, 0) \cup [7, 9]$
(b) $a \in (-\infty, 0) \cup [7, \infty]$
(c) $a \in (-\infty, 0) \cup [9, \infty)$
(d) None of these

34. x_1, x_2 are the roots of $x^2 - 3x + a = 0$ and x_3, x_4 are the roots of $x^2 - 12x + b = 0$. If x_1, x_2, x_3, x_4 form an increasing G.P. then ordered pair (a, b) is
(a) $(1, 16)$ (b) $(2, 32)$
(c) $(2, 16)$ (d) $(1, 32)$

35. If roots of $x^2 - (a - 3) x + a = 0$ are such that atleast one of them is greater than two, then
(a) $a \in [7, 9]$ (b) $a \in [7, \infty)$
(c) $a \in [9, \infty)$ (d) $a \in [7, 9)$

36. $f(x) = \dfrac{(x - b)(x - c)}{(x - a)}$, where a, b, c are distinct real numbers, will assume all real values provided
(a) c lies between a and b
(b) a lies between b and c
(c) b lies between a and c
(d) None of these

37. Let x_1 and x_2 are the roots of $ax^2 + bx + c = 0$ and $x_1 \cdot x_2 < 0$, $x + 1$ is non zero.
Roots of $x_1(x - x_2)^2 + x_2 (x - x_1)^2 = 0$ are
(a) negative
(b) real and of opposite signs
(c) positive
(d) non real

38. Consider the equation $x^2 + 2x - n = 0$, where $n \in N$ and $n \in [5, 100]$. Total number of different values of 'n' so that the given equation has integral roots is
(a) 4   (b) 8   (c) 3   (d) 6

39. $a, b, c \in R$ such that $abc \neq 0$. If x_1 is a root of $a^2x^2 + bx + c = 0$, x_2 is a root of $a^2x^2 - bx - c = 0$ and $x_1 > x_2 > 0$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root x_3 such that
(a) $x_1 > x_3 > x_2$ (b) $x_3 > x_1 > x_2$
(c) $x_1 > x_2 > x_3$ (d) None of these

40. If $\sin \alpha$, $\sin \beta$ and $\cos \alpha$ are in G.P., then roots of $x^2 + 2x \cot \beta + 1 = 0$ are always
(a) real
(b) real and negative
(c) greater than one
(d) non real

1. (b) : $ax^2 + 2bx + c = 0$

$\Rightarrow D = 4(b^2 - ac)$

Since 'b' is the A.M. of a and c

$\therefore \quad b^2 > ac \qquad [\because \text{A.M.} > \text{G.M.}]$

$\Rightarrow D > 0$

    Thus roots are real and distinct.

2. (a) : $x^2 + ax - (a^2 + 1) = 0$

Roots will be of opposite signs if $-(a^2 + 1) < 0$

$\Rightarrow a^2 + 1 > 0 \Rightarrow a \in R$

3. (b) : If a, b, c are odd integers, then

$ax^2 + bx + c = 0$ cannot have any rational root.

4. (d) : $x^2 - ax + a = 0$

$\therefore \quad D = a^2 - 4a \geq 0$

$\Rightarrow a \in (-\infty, 0] \cup [4, \infty)$

Both roots are greater than 2, if $f(2) > 0$

$\Rightarrow a \in (-\infty, 4)$ and $\dfrac{a}{2} > 2 \Rightarrow a > 4.$

Hence no such 'a' can be obtained.

5. (d) : $ax^2 + ax + 1 = 0$

$\Rightarrow D = a^2 - 4a \geq 0$

$\Rightarrow a \in (-\infty, 0] \cup [4, \infty)$

Since both roots are less than one.

i.e., $af(1) > 0, -\dfrac{a}{2a} < 1$ (which is always true as $a \neq 0$)

$\Rightarrow a(2a + 1) > 0$

$\Rightarrow a \in \left(-\infty, -\dfrac{1}{2}\right) \cup (0, \infty)$

$\Rightarrow a \in \left(-\infty, -\dfrac{1}{2}\right) \cup [4, \infty)$

6. (b) : $D = a^2 - 4(a + 1) = (a - 2)^2 - 8 = \lambda^2$ (say)

$\Rightarrow (a - 2)^2 - \lambda^2 = 8$

$\Rightarrow (a - 2 + \lambda)(a - 2 - \lambda) = 8$

Clearly, $a - 2 + \lambda$ and $a - 2 - \lambda$ are not odd.

Thus $a - 2 + \lambda = 4, \ a - 2 - \lambda = 2$

$\Rightarrow a = 5, \lambda = 1$

or $a - 2 + \lambda = 2, a - 2 - \lambda = 4$

$\Rightarrow a = 5, \lambda = -1$

or $a - 2 + \lambda = -4, a - 2 - \lambda = -2$

$\Rightarrow a = -1, \lambda = -1$

or $a - 2 + \lambda = -2, \ a - 2 - \lambda = -4$

$\Rightarrow a = -1, \lambda = 1$

Hence, $a = -1$ or 5.

7. (a) : $x^2 + 2ax + a < 0$ for $x \in [1, 2]$

$\Rightarrow 1 + 2a + a < 0, 4 + 4a + a < 0$

$\Rightarrow a < -\dfrac{1}{3}, a < -\dfrac{4}{5} \Rightarrow a \in \left(-\infty, -\dfrac{4}{5}\right)$

8. (d) : $x^2 + ax + 2 = 0$

$\Rightarrow D = a^2 - 8 \geq 0$

$\Rightarrow a \in (-\infty, -2\sqrt{2}] \cup [2\sqrt{2}, \infty)$

Since both the roots $\in (0, 3)$

$\therefore \quad f(0) > 0, f(3) > 0, 0 < -\dfrac{B}{2A} < 3$

$\Rightarrow 2 > 0, 9 + 3a + 2 > 0, \ 0 < -\dfrac{a}{2} < 3$

$\Rightarrow a > -\dfrac{11}{3}, 0 > a > -6$

$\Rightarrow a \in \left(-\dfrac{11}{3}, 0\right)$

Thus finally, $a \in \left(-\dfrac{11}{3}, -2\sqrt{2}\right]$

9. (b) : Roots of $x^2 + 3x + 5 = 0$ are non real. Thus given equations will have two common roots.

$\Rightarrow \dfrac{a}{1} = \dfrac{b}{3} = \dfrac{c}{5} = \lambda$ (say)

$\Rightarrow a + b + c = 9\lambda$

Thus minimum value of $a + b + c = 9$

10. (c): Let the roots be $x_1, -x_1, x_2$.

Then, $x_1 - x_1 + x_2 = a \Rightarrow x_2 = a$

Hence $x = a$ is a root of the given equation.

$\Rightarrow a^3 - a^3 + ab - c = 0$

$\Rightarrow ab = c$

11. (b) : $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$

$\Rightarrow D = (c + a - b)^2 - 4(b + a - c)(b - (a - c))$

    $= (c + a - b)^2 - 4(b^2 - (a - c)^2)$

    $= (c + a)^2 + b^2 - 2b(c + a) - 4b^2 + 4(a - c)^2$

    $= (-b)^2 + b^2 - 2b(-b) - 4b^2 + 4(a - c)^2$

                 $[\text{as } a + b + c = 0]$

    $= 4(a - c)^2 > 0$

12. (d) : $\sin\theta + \cos\theta = -\dfrac{b}{a}, \sin\theta \cdot \cos\theta = \dfrac{c}{a}$

$\Rightarrow \dfrac{b^2}{a^2} = 1 + 2 \cdot \dfrac{c}{a} \Rightarrow b^2 = a^2 + 2ac$

13. (d) : $x^2(a^2 - 1) + 2ax + 1 = 0$

$\Rightarrow D = 4a^2 - 4(a^2 - 1) = 4 > 0$

Both roots $\in (0, 1)$

$\therefore \quad (a^2 - 1)f(0) > 0, (a^2 - 1)f(1) > 0, 0 < -\dfrac{B}{2A} < 1$

$\Rightarrow a^2 - 1 > 0, (a^2 - 1)(a^2 + 2a) > 0, 0 < \dfrac{-a}{a^2 - 1} < 1$

$\Rightarrow a \in (-\infty, -2) \cup (1, \infty)$ and

$a \in \left(-\infty, \dfrac{-1 - \sqrt{5}}{2}\right) \cup \left(\dfrac{-1 + \sqrt{5}}{2}, \infty\right)$

Finally, $a \in (-\infty, -2) \cup (1, \infty)$

14. (d) : $x^2 + ax + a^2 + 6a < 0, \ \forall \ x \in [-1, 1]$

$\Rightarrow \ 1 - a + a^2 + 6a < 0$

$\Rightarrow \ a^2 + 5a + 1 < 0$

$\Rightarrow \ a \in \left(\dfrac{-5 - \sqrt{21}}{2}, \dfrac{-5 + \sqrt{21}}{2} \right)$

and $1 + a + a^2 + 6a < 0$

$\Rightarrow \ a^2 + 7a + 1 < 0$

$\Rightarrow \ a \in \left(\dfrac{-7 - \sqrt{45}}{2}, \dfrac{-7 + \sqrt{45}}{2} \right)$

$\Rightarrow \ a \in \left(\dfrac{-5 - \sqrt{21}}{2}, \dfrac{-5 + \sqrt{21}}{2} \right)$

15. (a) : $(a + 2)x^2 + 2ax - 1 = 0$

Sum of roots $= t - 1 - t - 1 = -2$

$\Rightarrow \ \dfrac{-2a}{(a + 2)} = -2 \ \Rightarrow \ a = a + 2$

which is not possible. Thus no such 'a' exists.

16. (c) : $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root

$\Rightarrow \ (bc - a^2)^2 = (ab - c^2)(ac - b^2)$

$\Rightarrow \ b^2c^2 + a^4 - 2a^2bc = a^2bc - ab^3 - ac^3 + b^2c^2$

$\Rightarrow \ a^4 + ab^3 + ac^3 = 3a^2bc$

$\Rightarrow \ \dfrac{a^3 + b^3 + c^3}{abc} = 3$

17. (a) : The equations $ax^3 + bx^2 + cx + d = 0$ and $(a + 1)x^3 + (b + 1)x^2 + (c + 1)x + (d + 1) = 0$ must be identical.

$\Rightarrow \ \dfrac{a + 1}{a} = \dfrac{b + 1}{b} = \dfrac{c + 1}{c} = \dfrac{d + 1}{d}$

$\Rightarrow \ a = b = c = d \neq 0$

18. (c): $\tan\theta + \sec\theta = -\dfrac{b}{a}, \ \tan\theta \cdot \sec\theta = \dfrac{c}{a}$

Since $\sec^2\theta - \tan^2\theta = 1$

$\therefore \ \sec\theta - \tan\theta = -\dfrac{a}{b}$

$\Rightarrow \ \sec\theta = -\dfrac{(a^2 + b^2)}{2ab}$ and $\tan\theta = \dfrac{(a^2 - b^2)}{2ab}$

$\Rightarrow \ \dfrac{(a^2 + b^2)(b^2 - a^2)}{4a^2b^2} = \dfrac{c}{a}$

$\Rightarrow \ b^4 - a^4 = 4acb^2$

$\Rightarrow \ a^4 = b^2(b^2 - 4ac)$

19. (b) : $x^3 + ax + 1 = 0, \ x^4 + ax^2 + 1 = 0$ have a common root.

Clearly this common root can't be equal to zero.

Multiplying first equation by x, we get

$x^4 + ax^2 + x = 0.$

Solving it with second equation we get $x = 1$.

Thus $x = 1$ is the common root.

$\Rightarrow \ 1 + a + 1 = 0 \Rightarrow a = -2$

20. (c): $b = \dfrac{2ac}{a + c}$ that means b is the H.M. of a and c.

Now, $ax^2 + 2bx + c = 0$

$\Rightarrow \ D = 4(b^2 - ac)$

Since G.M. of a and c is $\sqrt{ac}$

$\therefore \ ac > b^2 \qquad \qquad [\because \ \text{G.M.} > \text{H.M.}]$

$\Rightarrow \ D < 0$

21. (d) : Let $y = \dfrac{x^2 - x}{1 - ax}$

$\Rightarrow \ x^2 - x = y - axy \Rightarrow x^2 + x(ay - 1) - y = 0$

Since 'x' is real, therefore $(ay - 1)^2 + 4y \geq 0$

$\Rightarrow \ a^2y^2 + 2y(2 - a) + 1 \geq 0 \ \forall \ y \in R$

$\Rightarrow \ a^2 > 0, \ 4(2 - a)^2 - 4a^2 \leq 0$

$\Rightarrow \ 4 + a^2 - 4a - a^2 \leq 0 \Rightarrow a^2 > 0, \ 4 - 4a \leq 0$

$\Rightarrow \ 1 \leq a$

Hence $a \in [1, \infty)$

22. (a) : Clearly

$\dfrac{b}{a} > 0, \dfrac{c}{a} > 0, \dfrac{c}{b} > 0, \dfrac{a}{b} > 0, \dfrac{a}{c} > 0, \dfrac{b}{c} > 0$

Thus a, b, c all have same signs.

Also $b^2 \geq ac, \ c^2 \geq ab, \ a^2 \geq bc$

These last three inequalities can hold true simultaneously if and only if $a = b = c$.

23. (d) : $x^2 + a|x| + 1 = 0$

$\Rightarrow \ |x|^2 + a|x| + 1 = 0$

$\Rightarrow \ |x| = \dfrac{-a \pm \sqrt{a^2 - 4}}{2}$

Since $a < 0$ and $a \neq -2$

$\therefore \ \dfrac{-a + \sqrt{a^2 - 4}}{2}$ and $\dfrac{-a - \sqrt{a^2 - 4}}{2}$ both are positive.

Thus there are four roots if $a > -2$, else no real root.

24. (b) : $x^2 + ax + 1$ must divide $ax^3 + bx + c$.

Now, $\dfrac{ax^3 + bx + c}{x^2 + ax + 1}$

$= a(x - a) + \dfrac{(b - a + a^3)x + c + a^2}{x^2 + ax + 1}$

$\Rightarrow \ b - a + a^3 = 0, \ a^2 + c = 0$

25. (c): $x_1 + x_2 = -\dfrac{b}{a}$, $x_1 x_2 = \dfrac{c}{a}$

Now, $\dfrac{1}{x_1^2} + \dfrac{1}{x_2^2} = \dfrac{x_1^2 + x_2^2}{x_1^2 x_2^2}$

$$= \dfrac{(x_1 + x_2)^2 - 2x_1 x_2}{(x_1 x_2)^2}$$

$$= \left(\dfrac{x_1 + x_2}{x_1 x_2} \right)^2 - \dfrac{2}{x_1 x_2} = \dfrac{b^2}{c^2} - \dfrac{2a}{c}$$

We have, $\dfrac{b^2}{c^2} - \dfrac{2a}{c} = -\dfrac{b}{a}$

$\Rightarrow \quad \dfrac{b^2}{ac} - 2 = -\dfrac{bc}{a^2} \quad \Rightarrow \quad \dfrac{b^2}{ac} + \dfrac{bc}{a^2} = 2$

26. (b) : $(n - 2)x^2 + 8x + n + 4 > 0 \; \forall \; x \in R$

$\Rightarrow \quad 64 - 4(n - 2)(n + 4) < 0$

$\Rightarrow \quad 16 - (n^2 + 2n - 8) < 0$

$\Rightarrow \quad n^2 + 2n - 24 > 0$

$\Rightarrow \quad (n + 6)(n - 4) > 0$

$\Rightarrow \quad n > 4 \;\; \text{as} \;\; n \in N$

$\Rightarrow \quad n \geq 5$

27. (a) : $x^2 - 2ax + a^2 + a - 3 = 0$.

Both the roots are less than three. So, we must have

$D \geq 0 \; ; \; f(3) > 0; \; -\dfrac{B}{2A} < 3$

$\Rightarrow \quad 4a^2 - 4(a^2 + a - 3) \geq 0 \; ;$

$\quad 9 - 6a + a^2 + a - 3 > 0 \; ; \; a < 3$

$\Rightarrow \quad (a - 3) \leq 0 \, ; a^2 - 5a + 6 > 0 \, ; a < 3$

$\Rightarrow \quad a \leq 3 \, ; a < 2, \, a > 3 \, ; a < 3$

$\Rightarrow \quad a \in (-\infty, \; 2)$

28. (a) : $(x - a)(x - b) - f = (x - c)(x - d)$

$\Rightarrow \quad (x - c)(x - d) + f = (x - a)(x - b)$

Thus roots of $(x - c)(x - d) + f = 0$ are a and b.

29. (b) : $|x_1 - x_2| = 1$

$\Rightarrow \quad (x_1 + x_2)^2 - 4x_1 x_2 = 1$

$\Rightarrow \quad a^2 - 4b = 1$

$\Rightarrow \quad a^2 = 1 + 4b$.

30. (b) : Clearly the first equation has equal roots which are equal to $-\dfrac{b}{a}$ each. Thus it should also be the root of the second equation.

Thus, $d\left(-\dfrac{b}{a} \right)^2 + 2e\left(-\dfrac{b}{a} \right) + f = 0$

$\Rightarrow \quad \dfrac{d}{a} + \dfrac{f}{c} = 2\dfrac{eb}{ac} = 2\dfrac{e}{b}$

31. (d) : Let $t = x + \sqrt{x^2 + b^2}$

$\left(\sqrt{x^2 + b^2} - x \right)\left(\sqrt{x^2 + b^2} + x \right) = b^2$

$\Rightarrow \quad \sqrt{x^2 + b^2} - x = \dfrac{b^2}{t}$

$\Rightarrow \quad 2x = t - \dfrac{b^2}{t} \quad \Rightarrow \quad x = \dfrac{1}{2}\left(\dfrac{t^2 - b^2}{t} \right)$

Now, $y = 2(a - x)t = 2\left(a - \left(\dfrac{t^2 - b^2}{2t} \right) \right)t$

$\quad = (2at - t^2 + b^2) = b^2 - t^2 + 2at - a^2 + a^2$

$\quad = a^2 + b^2 - (t - a)^2$

$\Rightarrow \quad y \leq a^2 + b^2$

32. (c): Subtracting the given equations, we get

$(a - b)x + c(b - a) = 0$

$\Rightarrow \quad x = c$ is the common root.

Thus roots of $x^2 + ax + bc = 0$ are b and c and that of $x^2 + bx + ca = 0$ are c and a

Also, $b + c = -a$.

Hence the required equation is

$x^2 - (a + b)x + ab = 0$

i.e., $x^2 + cx + ab = 0$

33. (c): $x^2 - (a - 3)x + a = 0$

$D = (a - 3)^2 - 4a = a^2 - 10a + 9$

Now $D \geq 0$

$\Rightarrow$ $(a - 9)(a - 1) \geq 0$

$\Rightarrow$ $a \in (-\infty, 1] \cup [9, \infty)$

Case 1 : When both roots are positive

$\quad D \geq 0, \; a - 3 > 0, \; a > 0$

$\Rightarrow$ $D \geq 0, \; a > 0, \; a > 3$

$\Rightarrow$ $a \in [9, \infty)$

Case 2 : When exactly one root is positive

$\Rightarrow$ $a < 0 \Rightarrow a \in (-\infty, 0)$

Thus finally $a \in (-\infty, 0) \cup [9, \infty)$

34. (b) : Let $x_1 = A, \; x_2 = AR, \; x_3 = AR^2, \; x_4 = AR^3$

Thus we have,

$\quad x_1 + x_2 = 3$

$\Rightarrow$ $A(1 + R) = 3$

and $\; x_1 x_2 = a \Rightarrow A^2 R = a$

Also, $\; x_3 + x_4 = 12$

$\Rightarrow$ $AR^2(1 + R) = 12 \;$ and $\; x_3 x_4 = b$

$\Rightarrow$ $A^2 R^5 = b$

On solving, we get $R^2 = 4$

$\Rightarrow$ $R = 2$ (as it is an increasing G.P.)

Now, $\; A = \dfrac{3}{3} = 1$

$\Rightarrow$ $a = 2, \; b = 32$

Thus required ordered pair is $(2, 32)$

35. (c) : $x^2 - (a - 3)x + a = 0$

$\Rightarrow$ $D = (a - 3)^2 - 4a = a^2 - 10a + 9$

$\quad\quad = (a - 1)(a - 9)$

Case 1 : When both roots are greater than 2.

$D \geq 0, \; f(2) > 0, \; -\dfrac{B}{2A} > 2$

$\Rightarrow$ $(a - 1)(a - 9) \geq 0 ; 4 - (a - 3)2 + a > 0 ; \dfrac{a - 3}{2} > 2$

$\Rightarrow$ $a \in (-\infty, 1] \cup [9, \infty) ; a < 10 ; a > 7$

$\Rightarrow$ $a \in [9, 10)$

Case 2 : One root is > 2 and other is less than or equal to 2. Thus

$f(2) \leq 0, \; D \geq 0$

$\Rightarrow$ $4 - (a - 3)2 + a \leq 0, \; (a - 1)(a - 9) \geq 0$

$\Rightarrow$ $a \geq 10, \; a \leq 1 \;$ or $\; a \geq 9.$

$\Rightarrow$ $a \geq 10 \Rightarrow a \in [10, \infty)$

Finally $a \in [9, 10) \cup [10, \infty)$

$\Rightarrow$ $a \in [9, \infty)$

36. (b) : Let $y = \dfrac{(x - b)(x - c)}{x - a}$

$\Rightarrow$ $x^2 - (b + c)x + bc = yx - ya$

$\Rightarrow$ $x^2 - (b + c + y)x + bc + ya = 0$

Since $\; x \in R$, therefore

$(b + c + y)^2 - 4(bc + ya) \geq 0$

$\Rightarrow$ $y^2 + (b + c)^2 + 2y(b + c) - 4bc - 4ya \geq 0$

$\Rightarrow$ $y^2 + 2y(b + c - 2a) + (b - c)^2 \geq 0$

Since y should take all real values, therefore

$4(b + c - 2a)^2 - 4(b - c)^2 \leq 0$

$\Rightarrow$ $(b + c - 2a)^2 - (b - c)^2 \leq 0$

$\Rightarrow$ $(2b - 2a)(2c - 2a) \leq 0$

$\Rightarrow$ $(a - b)(a - c) \leq 0$

Hence a should lie between b and c.

37. (b) : $x_1(x - x_2)^2 + x_2(x - x_1)^2 = 0$

$\Rightarrow$ $x^2(x_1 + x_2) - 4x\,x_1 x_2 + x_1 x_2(x_1 + x_2) = 0$

$D = 16(x_1 x_2)^2 - 4x_1 x_2 \cdot (x_1 + x_2)^2 > 0$ as $x_1 x_2 < 0$

Product of roots = $x_1 x_2 < 0$

Thus roots are real and of opposite signs.

38. (b) : $x^2 + 2x - n = 0$

$\Rightarrow$ $(x + 1)^2 = n + 1$

$\Rightarrow$ $x = -1 \pm \sqrt{n + 1}$

Thus $n + 1$ should be a perfect square.

Since $\; n \in [5, \; 100]$, therefore

$\quad n + 1 \in [6, \; 101]$

Number of perfect squares from 1 to 100 is 10. Thus n can take $10 - 2$ i.e., 8 different values.

39. (a) : $a^2 x_1^2 + bx_1 + c = 0, \; a^2 x_2^2 - bx_2 - c = 0$

Let $f(x) = a^2 x^2 + 2bx + 2c$

$\Rightarrow$ $f(x_1) = a^2 x_1^2 + 2bx_1 + 2c = -a^2 x_1^2$

and $f(x_2) = a^2 x_2^2 + 2bx_2 + 2c = 3a^2 x_2^2$

$\Rightarrow$ $f(x_1) \cdot f(x_2) = (3a^2 x_2^2)(-a^2 x_1^2) < 0$

Thus one root of $\; a^2 x^2 + 2bx + 2c = 0$ will lie between x_1 and x_2.

40. (a) : $\sin^2\beta = \sin\alpha \cdot \cos\alpha$

$\Rightarrow$ $2 \sin^2\beta = \sin 2\alpha$

$\Rightarrow$ $\sin^2\beta \leq \dfrac{1}{2} \Rightarrow \text{cosec}^2\beta \geq 2$

Now discriminant of $x^2 + 2x \cot\beta + 1 = 0$ is

$4\cot^2\beta - 4 = 4(\cot^2\beta - 1) = 4(\text{cosec}^2\beta - 2)$

$\Rightarrow$ $D \geq 0$

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE(Main & Advanced) Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE(Main & Advanced). In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. BC is latus rectum of a parabola $y^2 = 4ax$ and A is its vertex. The minimum length of projection of BC on a tangent drawn in portion BAC is
   (a) $\sqrt{2}a$ (b) $2\sqrt{2}a$
   (c) $2a$ (d) $3\sqrt{2}a$

2. Least value of the expression
   $\dfrac{1}{2bx - (x^2 + b^2 + \sin^2 x)}, x \in [-1, 0], b \in [2, 3]$ is
   (a) $\dfrac{1}{4}$ (b) $-\dfrac{1}{4}$
   (c) $-\dfrac{1}{8 + \sin^2 1}$ (d) none of these

3. If in a right angled triangle ABC, $4\sin A \cos B - 1 = 0$ and $\tan A$ is real, then
   (a) angles are in A.P. (b) angles are in G.P.
   (c) angles are in H.P. (d) none of these

4. If $|z| = 2$ and $\dfrac{z_1 - z_3}{z_2 - z_3} = \dfrac{z - 2}{z + 2}$, then z_1, z_2, z_3 will be vertices of a
   (a) equilateral triangle
   (b) acute angled triangle
   (c) right angled triangle
   (d) none of these

5. If $a^2 + b^2 - c^2 - 2ab = 0$, then the point(s) of concurrency of family of straight lines $ax + by + c = 0$ lie(s) on the line
   (a) $y = x$ (b) $y = x + 1$
   (c) $y = -x$ (d) $x + y = 1$

6. If $f''(x) > 0 \; \forall \; x \in R, f'(3) = 0$
   and $g(x) = f(\tan^2 x - 2\tan x + 4), 0 < x < \dfrac{\pi}{2}$,
   then $g(x)$ is increasing in
   (a) $\left(0, \dfrac{\pi}{4}\right)$ (b) $\left(\dfrac{\pi}{6}, \dfrac{\pi}{3}\right)$

   (c) $\left(0, \dfrac{\pi}{3}\right)$ (d) $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

7. A vector $\vec{r}$ is equally inclined with the vectors $\vec{a} = \cos\theta\hat{i} + \sin\theta\hat{j}$, $\vec{b} = -\sin\theta\hat{i} + \cos\theta\hat{j}$ and $\vec{c} = \hat{k}$, then angle between $\vec{r}$ and $\vec{a}$ is
   (a) $\cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ (b) $\cos^{-1}\left(\dfrac{1}{3}\right)$
   (c) $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ (d) $\dfrac{\pi}{2}$

8. If the roots of the equation $x^2 + ax + b = 0$ are c and d, then one of the roots of the equation $x^2 + (2c + a) x + c^2 + ac + b = 0$ is
   (a) c (b) $d - c$ (c) $2c$ (d) $2d$

9. Let f be a differentiable function satisfying the condition $f\left(\dfrac{x}{y}\right) = \dfrac{f(x)}{f(y)}$ for all $x, y \, (\neq 0) \in R$ and $f(y) \neq 0$. If $f'(1) = 2$, then $f'(x)$ is equal to
   (a) $2f(x)$ (b) $\dfrac{f(x)}{x}$ (c) $2xf(x)$ (d) $\dfrac{2f(x)}{x}$

10. The number of solutions of the equation $\cos^{-1} x + \cos^{-1}\sqrt{1 - x^2} = \pi$ is
   (a) 1   (b) 2
   (c) 0   (d) none of these

SOLUTIONS

1. (b): Let tangent at $P(at^2, 2at)$ makes an angle θ with x-axis, then $\tan\theta = \dfrac{1}{t}$
   Projection of BC on tangent = $BC\sin\theta$
   $= \dfrac{4a}{\sqrt{1 + t^2}} \geq 2a\sqrt{2}$ (as $-1 \leq t \leq 1$).

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

2. **(b)**: Given expression will have least value if
$2bx - [x^2 + b^2 + \sin^2 x]$ is maximum

$x^2 + b^2 + \sin^2 x - 2bx$ is minimum

$(x - b)^2 + \sin^2 x$ is minimum

Now $|x - b|$ and $|\sin x|$ are minimum if $x = 0$, $b = 2$
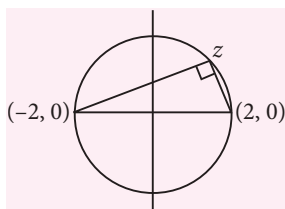
So, least value is $-\dfrac{1}{4}$

3. **(a)** : Since, $4 \sin A \cos B = 1$, so A and B can not be $\dfrac{\pi}{2}$

[As if $B = \dfrac{\pi}{2}$, then $\cos B = 0$ and if $A = \dfrac{\pi}{2}$, $\tan A$

is not defined]

$C = \dfrac{\pi}{2}, B = \dfrac{\pi}{2} - A \Rightarrow 4 \sin A \cos\left(\dfrac{\pi}{2} - A\right) = 1$

$\sin^2 A = \dfrac{1}{4} \Rightarrow \sin A = \dfrac{1}{2} \Rightarrow A = \dfrac{\pi}{6} \Rightarrow B = \dfrac{\pi}{3}$

So angles are $\dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{\pi}{2}$ which are in A.P.

4. **(c)** : Clearly $\text{Arg}\left(\dfrac{z-2}{z+2}\right) = \pm\dfrac{\pi}{2}$

$\Rightarrow \text{Arg}\left(\dfrac{z_1 - z_3}{z_2 - z_3}\right) = \pm\dfrac{\pi}{2}$

So z_1, z_2, z_3 will be the vertices of a right angled triangle.

5. **(c)** : $(a - b)^2 - c^2 = 0$

$\Rightarrow (a - b - c)(a - b + c) = 0$

If $a - b = c \Rightarrow ax + by + a - b = 0$

$\Rightarrow (x + 1)a + b(y - 1) = 0$

$\Rightarrow x = -1, y = 1$

If $-a + b = c \Rightarrow ax + by + b - a = 0$

$\Rightarrow (x - 1)a + (y + 1)b = 0$

$\Rightarrow (x - 1) + (y + 1)\dfrac{b}{a} = 0$

$\Rightarrow x = 1, y = -1$

Equation of line passing through both points $(-1, 1)$ and $(1, -1)$ is $y = -x$.

6. **(d)**: $g'(x) = f'((\tan x - 1)^2 + 3)(2\tan x - 2)\sec^2 x$

Since $f''(x) > 0 \Rightarrow f'(x)$ is increasing

So $f'((\tan x - 1)^2 + 3) > f'(3) = 0$

$\forall x \in \left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

Also $(\tan x - 1) > 0 \ \forall x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

So, $g(x)$ is increasing in $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$.

7. **(c)** : Since $\vec{a}\cdot\vec{b} = 0 = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular.

Now $\vec{r} = t(\vec{a} + \vec{b} + \vec{c}) = t((\cos\theta - \sin\theta)i$
$+ (\cos\theta + \sin\theta)\hat{j} + \hat{k})$

Let angle between $\vec{r}$ and $\vec{a}$ be α, then

$\cos\alpha = \dfrac{\vec{r}\cdot\vec{a}}{|\vec{r}||\vec{a}|} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \alpha = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

8. **(b)**: Let $f(x) = x^2 + ax + b$, then
$x^2 + (2c + a)x + c^2 + ac + b = f(x + c)$

Thus roots of $f(x + c) = 0$ will be 0, $(d - c)$.

9. **(d)**: We have

$f\left(\dfrac{x}{y}\right) = \dfrac{f(x)}{f(y)}$

Replacing x and y both by 1,

$f(1) = \dfrac{f(1)}{f(1)} = 1$

$f'(x) = \lim_{h\to 0} \dfrac{f(x + h) - f(x)}{h}$

$= f(x)\lim_{h\to 0}\left\{\dfrac{\dfrac{f(x+h)}{f(x)} - 1}{h}\right\} = f(x)\lim_{h\to 0}\dfrac{f\left(\dfrac{x+h}{x}\right) - 1}{h}$

$= \dfrac{f(x)}{x}\lim_{h\to 0}\dfrac{f\left(1 + \dfrac{h}{x}\right) - f(1)}{h/x}$

$= \dfrac{f(x)}{x}f'(1) = \dfrac{2f(x)}{x} \qquad (\because f'(1) = 2)$

10. **(a)**: $\cos^{-1}\sqrt{1 - x^2} = \pi - \cos^{-1}x = \cos^{-1}(-x)$
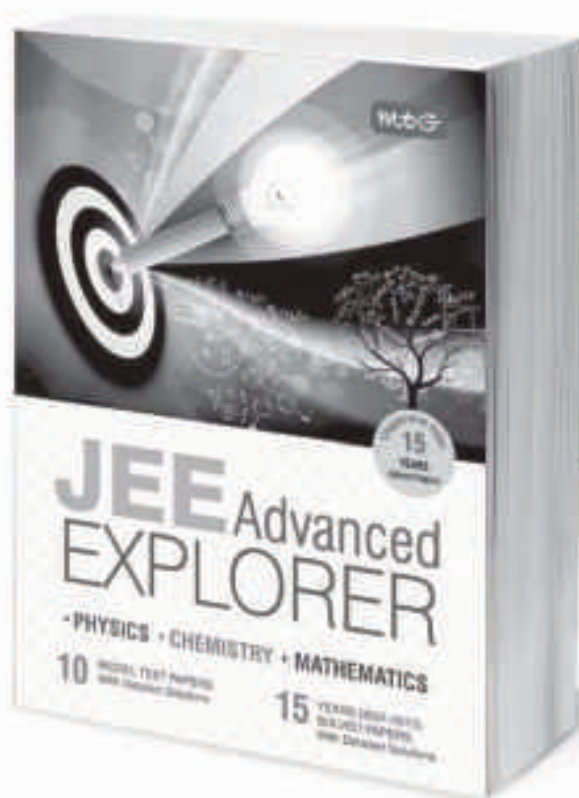
$\Rightarrow \sqrt{1 - x^2} = -x \Rightarrow x < 0$

Squaring, $1 - x^2 = x^2 \Rightarrow x = \pm\dfrac{1}{\sqrt{2}}$

Since $x < 0$, $x = -\dfrac{1}{\sqrt{2}}$

Y?U ASK
WE ANSWER

1. The interval in which the function $f(x) = \sin(\log_e x) - \cos(\log_e x)$ strictly increases is

Kishen Gowda, A.P.

Ans. $f(x) = \sin(\log_e x) - \cos(\log_e x)$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{\cos(\log_e x) + \sin(\log_e x)}{x}$$

For critical points, we put $f'(x) = 0$

$\Rightarrow \quad \cos(\log_e x) + \sin(\log_e x) = 0$

$\Rightarrow \quad \tan(\log_e x) = -1 = \tan\dfrac{3\pi}{4}$

$\Rightarrow \quad \log_e x = n\pi + \dfrac{3\pi}{4},\, n \in Z$

$\Rightarrow \quad \log_e x = ..., \dfrac{-9\pi}{4}, \dfrac{-5\pi}{4}, \dfrac{-\pi}{4}, \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \dfrac{11\pi}{4}, ...$

$\Rightarrow \quad x = ... e^{\frac{-9\pi}{4}}, e^{\frac{-5\pi}{4}}, e^{\frac{-\pi}{4}}, e^{\frac{3\pi}{4}}, e^{\frac{7\pi}{4}}, ...$
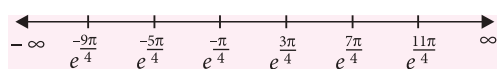
So, we get $f'(x) > 0$,

when $x \in .... \cup \left(e^{\frac{-9\pi}{4}}, e^{\frac{-5\pi}{4}}\right) \cup \left(e^{\frac{-\pi}{4}}, e^{\frac{3\pi}{4}}\right)$

$$\cup \left(e^{\frac{7\pi}{4}}, e^{\frac{11\pi}{4}}\right) \cup ....$$

$\Rightarrow$ f is strictly increasing in

$$\left(e^{2n\pi - \frac{\pi}{4}}, e^{2n\pi + \frac{3\pi}{4}}\right), n \in Z$$

2. If m things are distributed among 'a' men and 'b' women, show that the probability that the number of things received by men is odd, is

$$\frac{1}{2} \frac{\{(b+a)^m - (b-a)^m\}}{(b+a)^m}$$

Rizul Gautam, H.P.

Ans. Let p be the probability that any one thing is received by a man and q be the probability that any one thing is received by a woman.

$\therefore \quad p = \dfrac{a}{a+b}$ and $q = \dfrac{b}{a+b}$

Clearly, $p + q = 1$ *i.e.*, $q = 1 - p$

Out of m things, if r are received by a man then the rest $(m - r)$ will be received by women.

The probability for this to happen is given by

$P(r) = {}^mC_r\, p^r q^{m-r} \qquad (r = 0, 1, ..., m)$

The probability P that odd number of things are received by men is given by

$P = P(1) + P(3) + P(5) + ...$

$= {}^mC_1\, pq^{m-1} + {}^mC_3\, p^3 q^{m-3} + {}^mC_5\, p^5 q^{m-5} + ...$

...(i)

We know that

$(q + p)^m = q^m + {}^mC_1 q^{m-1} p + {}^mC_2 q^{m-2} p^2 + ... + p^m$

...(ii)

and $(q - p)^m = q^m - {}^mC_1 q^{m-1} p + {}^mC_2 q^{m-2} p^2 - ....$
$\qquad\qquad\qquad + (-1)^m \cdot p^m$ ...(iii)

Subtracting (iii) from (ii), we get

$(q + p)^m - (q - p)^m$

$= 2 \{{}^mC_1 q^{m-1} p + {}^mC_3 q^{m-3} p^3 + ...\} = 2P$

[From (i)]

$\therefore \quad P = \dfrac{1}{2}\{(q + p)^m - (q - p)^m\}$

$= \dfrac{1}{2}\left\{1 - \left(\dfrac{b-a}{b+a}\right)^m\right\} = \dfrac{1}{2}\left\{\dfrac{(b+a)^m - (b-a)^m}{(b+a)^m}\right\}$

3. Evaluate : $\displaystyle\int_0^{\pi/6} \frac{\sqrt{3\cos 2x - 1}}{\cos x}\, dx$

Pratyush Sinha, New Delhi

Ans. Let $I = \displaystyle\int_0^{\pi/6} \frac{\sqrt{3\cos 2x - 1}}{\cos x}\, dx$

$= \displaystyle\int_0^{\pi/6} \frac{\sqrt{2 - 6\sin^2 x}}{\cos^2 x} \cdot \cos x\, dx$

$$= \int_0^{\pi/6} \frac{\sqrt{2}\sqrt{1-3\sin^2 x}}{1-\sin^2 x} \cdot \cos x\, dx$$

Put $\sin x = \dfrac{1}{\sqrt{3}}\sin\theta \Rightarrow \cos x\, dx = \dfrac{1}{\sqrt{3}}\cos\theta\, d\theta$.

Also, when $x = 0$, $\theta = 0$ and $x = \dfrac{\pi}{6}$, $\theta = \dfrac{\pi}{3}$.
Hence, we have

$$I = \int_0^{\pi/3} \frac{\sqrt{2}\sqrt{1-\sin^2\theta}}{1-\dfrac{\sin^2\theta}{3}} \cdot \frac{1}{\sqrt{3}}\cos\theta\, d\theta$$

$$= \int_0^{\pi/3} \frac{\sqrt{6}\cos^2\theta}{3-\sin^2\theta}\, d\theta = \sqrt{6}\int_0^{\pi/3} \frac{3-\sin^2\theta-2}{3-\sin^2\theta}\, d\theta$$

$$= \sqrt{6}\int_0^{\pi/3} 1\, d\theta - \sqrt{6}\int_0^{\pi/3} \frac{2}{3-\left(\dfrac{1-\cos 2\theta}{2}\right)}\, d\theta$$

$$= \sqrt{6}\left(\frac{\pi}{3}\right) - 4\sqrt{6}\int_0^{\pi/3} \frac{d\theta}{5+\cos 2\theta}$$

Now, putting $\tan\theta = t$, $d\theta = \dfrac{dt}{1+t^2}$

and $\cos 2\theta = \dfrac{1-t^2}{1+t^2}$

When $\theta = 0$, $t = 0$ and $\theta = \dfrac{\pi}{3}$, $t = \sqrt{3}$
We have

$$I = \frac{\sqrt{2}\pi}{\sqrt{3}} - 4\sqrt{6}\int_0^{\sqrt{3}} \frac{dt}{6+4t^2}$$

$$= \frac{\sqrt{2}\pi}{\sqrt{3}} - \sqrt{6}\cdot\frac{1}{\sqrt{3/2}}\left[\tan^{-1}\left(\frac{t}{\sqrt{3/2}}\right)\right]_0^{\sqrt{3}}$$

$$= \frac{\sqrt{2}\pi}{\sqrt{3}} - 2\tan^{-1}\sqrt{2}.$$

■■

QUANTITATIVE APTITUDE

1. Let a, b, c, d and e be integers such that $a = 6b = 12\,c$, and $2b = 9d = 12\,e$. Then which of the following pair contains a number that is not an integer?

   (a) $\left(\dfrac{a}{27}, \dfrac{b}{e} \right)$ (b) $\left(\dfrac{a}{36}, \dfrac{c}{e} \right)$

   (c) $\left(\dfrac{a}{12}, \dfrac{bd}{18} \right)$ (d) $\left(\dfrac{a}{6}, \dfrac{c}{d} \right)$

2. There are 12 towns grouped into four zones with three towns per zone. It is intended to connect the towns with telephone lines such that every two towns are connected with three direct lines if they belong to the same zone, and with only one direct line otherwise. How many direct telephone lines are required?

   (a) 72   (b) 90   (c) 96   (d) 144

3. In how many ways can 10 examination papers be arranged so that the best and the worst papers never come together?

   (a) $9! \times 2!$ (b) $7! \times 2!$ (c) $8 \times 9!$ (d) $10!$

4. Mr. Palkiwala was to earn ₹ 300 and a free holiday for seven weeks' work. He worked for only 4 weeks and earned ₹ 30 and the free holiday. What is the monetary value of the holiday?

   (a) ₹ 300 (b) ₹ 330 (c) ₹ 360 (d) ₹ 420

5. Let $n(>1)$ be a composite integer such that $\sqrt{n}$ is not an integer. Consider the following statements

   P : n has a perfect integer-valued divisor which is greater than 1 and less than $\sqrt{n}$.

   Q : n has a perfect integer-valued divisor which is greater than $\sqrt{n}$ but less than n.

   Then,

   (a) Both P and Q are false
   (b) P is true but Q is false
   (c) P is false but Q is true
   (d) Both P and Q are true

6. Let S_1 be a square of side a. Another square S_2 is formed by joining the mid-points of the sides of S_1. The same process is applied to S_2 to form yet another square S_3, and so on. If A_1, A_2, A_3,... be the areas and P_1, P_2, P_3,..... be the perimeters of S_1, S_2, S_3, ..., respectively, then the ratio $\dfrac{P_1 + P_2 + P_3 + ...}{A_1 + A_2 + A_3 + ...}$ equals

   (a) $\dfrac{2(1+\sqrt{2})}{a}$ (b) $\dfrac{2(2-\sqrt{2})}{a}$

   (c) $\dfrac{2(2+\sqrt{2})}{a}$ (d) $\dfrac{2(1-\sqrt{2})}{a}$

7. There are two fields, area of one being twice as that of the other. Certain workers have been given the task of ploughing these fields. All of these workers work on the larger field for half day and for remaining half day, half of the workers work on the larger field, while the remaining work on the smaller field. At the end of the day, the larger field is entirely ploughed. While some part of the smaller field is left and it is completed by one worker using the next day. The number of workers working on the fields initially were (Assume that the number of workers required to plough a field is proportional to its area).

   (a) 4   (b) 15   (c) 8   (d) 20

8. The infinite sum $1 + \dfrac{4}{7} + \dfrac{9}{7^2} + \dfrac{16}{7^3} + \dfrac{25}{7^4} + ...$ equals

   (a) $\dfrac{27}{14}$ (b) $\dfrac{21}{13}$ (c) $\dfrac{49}{27}$ (d) $\dfrac{256}{147}$

9. A student was promised (by a generous teacher, of course) a prize scheme as follows.

   For the first problem solved correctly he would receive 1 paisa, for the second problem solved correctly he would receive 2 paise, for the third 4 paise and so on. The student turned out to be sharp enough and forced the teacher to dish out ₹ 40.95. The number of problems solved correctly were

   (a) 12   (b) 13   (c) 14   (d) 15

10. Let $a_1, a_2, ..., a_{10}$ be in A.P. and $h_1, h_2,..., h_{10}$ be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then a_4h_7 is :
(a) 2   (b) 3   (c) 5   (d) 6

11. *ABCD* is a square, inside which 4 circles with radius 1 cm each, as shown in figure. What is the area of the shaded region?
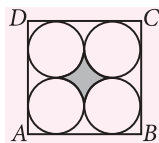
(a) $(2\pi - 3)$ cm^2 (b) $(4 - \pi)$ cm^2
(c) $(16 - 4\pi)$ cm^2 (d) None of these

12. Eleven years earlier the average age of a family of 4 members was 28 years. Now the age of the same family with six members is yet the same, even when 2 children were born in this period. If they belong to the same parents and the age of the first child at the time of the birth of the younger child was same as there were total family members just after the birth of the youngest member of this family, then the present age of the youngest member of the family is :
(a) 3 years   (b) 5 years
(c) 6 years   (d) None of these

13. ICICI lent ₹ 1 lakh to captain Ram Singh @ 6 % per annum of simple interest for 10 years period. Meanwhile ICICI offered a discount in rate of interest for armed forces. Thus the rate of interest ICICI decreased to 4%. In this way Ram Singh had to pay total amount 1.48 lakh.

After how many years Ram Singh got the discount in rate of interest?
(a) 3 years   (b) 4 years
(c) 6 years   (d) 5 years

14. On January 1, 2004 two new societies S_1 and S_2 are formed, each with n members. On the first day of each subsequent month, S_1 adds b members while S_2 multiplies its current number of members by a constant factor r. Both the societies have the same number of members on July 2, 2004. If $b = 10.5\ n$, what is the value of r?
(a) 2.0   (b) 1.9   (c) 1.8   (d) 1.7

15. A jogging park has two identical circular tracks touching each other, and a rectangular track enclosing the two circles. The edges of the rectangles are tangential to the circles. Two friends, *A* and *B*, start jogging simultaneously from the point where one of the circular tracks touches the smaller side of the rectangular track, *A* jogs along the rectangular track, while *B* jogs along the two circular tracks in a figure of eight. Approximately, how much faster

than *A* does *B* have to run, so that they take the same time to return to their starting point?
(a) 3.88% (b) 4.22% (c) 4.44% (d) 4.72%

16. A survey was conducted of 100 people to find out whether they had read recent issues of Golmal, a monthly magazine. The summarized information regarding readership in 3 months is given below :

Only September : 18; September but not August : 23; September and July : 8;

September : 28; July : 48, July and August : 10;

None of the three months : 24.

What is the number of surveyed people who have read exactly two consecutive issues (out of the three)?
(a) 7   (b) 9   (c) 12   (d) 14

17. A group of 630 children is arranged in rows for a group photograph session. Each row contains three fewer children than the row in front of it. What number of rows is not possible?
(a) 3   (b) 4   (c) 5   (d) 6

18. The number of employees in Obelix Menhir company is a prime number and is less than 300. The ratio of the number of employees who are graduates and above, to that of employees who are not, can possibly be :
(a) 101 : 88   (b) 87 : 100
(c) 110 : 111   (d) 97 : 84

19. A man purchased m oranges at x a rupee and n oranges at y a rupee. He mixed them together and sold them at z a rupee. What is his per cent loss or gain?

(a) $\left[\dfrac{(m-n)}{(my+nx)z} - 1\right] \times 100\%$ Loss

(b) $\left[\dfrac{(m+n)y}{(mx+ny)} + 1\right] \times 100\%$ Gain

(c) $\left[\dfrac{(m+n)xy}{(my+nx)z} - 1\right] \times 100\%$ Gain

(d) None of these

20. In three vessels each of 10 litres capacity, mixture of milk and water is filled. The ratios of milk and water are 2 : 1, 3 : 1 and 3 : 2 in the three respective vessels. If all the three vessels are emptied into a single vessel, find the proportion of milk and water in the mixture.
(a) 121 : 59   (b) 20 : 47
(c) 11 : 59   (d) 121 : 47

21. The ratio between the present ages of A and B is 5 : 3 respectively. The ratio between A's age 4 years ago and B's age 4 years hence is 1 : 1. What is the ratio between A's age 4 years hence and B's age 4 years ago?

(a) 2 : 1   (b) 3 : 1   (c) 2 : 3   (d) 1 : 2

22. A student is allowed to select at most n books from a collection of $(2n + 1)$ books. If the total number of ways in which he can select atleast one book is 63, find the value of n.

(a) 5   (b) 6   (c) 3   (d) 7

23. The following line-graph gives the annual profit earned by a company during the period 1995-2000. Study the line graph and answer the question that follows.

Percent Profit Earned by a Company over the Years

$$\% \text{ Profit} = \frac{\text{Income} - \text{Expenditure}}{\text{Expenditure}} \times 100$$
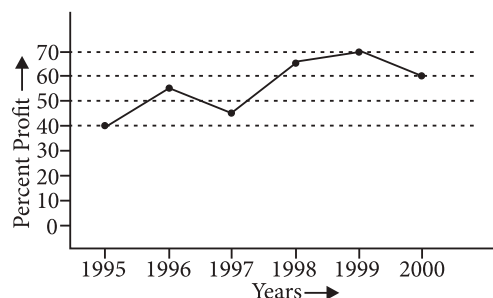
If the expenditures in 1996 and 1999 are equal, then the ratio of the income in 1996 and 1999 respectively is

(a) 1 : 1   (b) 2 : 3
(c) 31 : 34   (d) 13 : 14

24. Two cars P and Q start at the same time from A and B which are 120 km apart. If the two cars travel in opposite directions, they meet after one hour and if they travel in same direction (from A towards B), then P meets Q after 6 hours. What is the speed of car P?

(a) 60 km/hr   (b) 70 km/hr
(c) 120 km/hr   (d) Data inadequate

25. A student secures 90%, 60% and 54% marks in test papers with 100, 150 and 200 respectively as maximum marks. The percentage of his aggregate is

(a) 64   (b) 68
(c) 70   (d) None of these

ANSWER KEY									
1.	(d)	**2.**	(b)	**3.**	(c)	**4.**	(b)	**5.**	(d)
6.	(c)	**7.**	(c)	**8.**	(c)	**9.**	(a)	**10.**	(d)
11.	(b)	**12.**	(a)	**13.**	(b)	**14.**	(a)	**15.**	(d)
16.	(b)	**17.**	(d)	**18.**	(d)	**19.**	(c)	**20.**	(a)
21.	(b)	**22.**	(c)	**23.**	(c)	**24.**	(b)	**25.**	(a)

MATHS MUSING
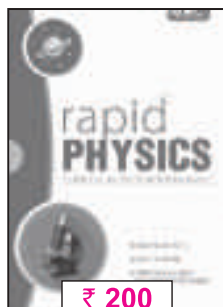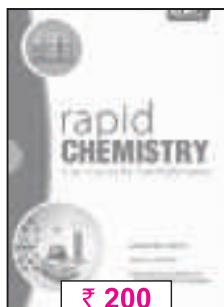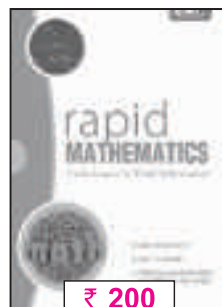
1. **(c) :** $\sqrt{x} - 1 = t^2 \Rightarrow I = \int_0^{\sqrt{3}} \tan^{-1} t \, d(t^2 + 1)^2$

   $= \dfrac{16\pi}{3} - 2\sqrt{3}.$

2. **(c) :** $z = \sin(\alpha + \beta) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$

   $z^2 = x^2 + y^2 - 2x^2 y^2 + 2xy\sqrt{1 - x^2}\sqrt{1 - y^2}$

   $\cos(\alpha + \beta) = \sqrt{1 - x^2}\sqrt{1 - y^2} - xy$

   $= \dfrac{z^2 - x^2 - y^2 + 2x^2 y^2}{2xy} - xy = \dfrac{z^2 - x^2 - y^2}{2xy}$

3. **(c) :** The plane ABC is $x + \dfrac{y}{2} + \dfrac{z}{3} = 1$

   If $H(\alpha, \beta, \gamma)$ is the orthocentre, then $AH \perp BC$

   $BH \perp CA$ ∴ $\alpha = 2\beta = 3\gamma$

   H lies in the plane ABC

   $\Rightarrow \dfrac{\alpha}{1} + \dfrac{\alpha}{4} + \dfrac{\alpha}{9} = 1$, $\alpha = \dfrac{36}{49}$, $H = \left(\dfrac{36}{49}, \dfrac{18}{49}, \dfrac{12}{49}\right)$

   The circumcentre, orthocentre, centroid G are collinear and $G = \left(\dfrac{1}{3}, \dfrac{2}{3}, 1\right)$

   The d.r's of HG are proportional to 59, –44, –111

   $a = 59$, $b = -44$, $a + b = 15$.

4. **(a) :** $\dfrac{1}{2} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} < \dfrac{3}{c}$

   $\Rightarrow c = 3, 4, 5$

   $c = 3 \Rightarrow (a - 6)(b - 6) = 36$ gives the triples
   (42, 7, 3), (24, 8, 3), (18, 9, 3), (15, 10, 3)

   $c = 4 \Rightarrow (a - 4)(b - 4) = 16$ gives (20, 5, 4), (12, 6, 4)

   $c = 5 \Rightarrow$ no triple

   ∴ $N = 4 + 2 = 6$.

5. **(c) :** Differentiating $(1 + px + x^2)^n = \sum_{r=0}^{2n} a_r x^r$ ...(i)

   $n(1 + px + x^2)^{n-1}(p + 2x) = \sum_{r=0}^{2n} r a_r x^{r-1}$ ...(ii)

   Setting $x = 1$ in (i) and (ii), we get

   $(p + 2)^n = \sum_{r=0}^{2n} a_r$, $n(p + 2)^n = \sum_{r=0}^{2n} r a_r$

   ∴ $\sum_{r=0}^{2n} (2r + 1) a_r = (2n + 1)(p + 2)^n$

6. **(c) :** $\lim_{x \to 1} \dfrac{f(x)}{(x - 1)^2} = 1 \Rightarrow f(1) = 0$

   $\lim_{x \to 1} \dfrac{f'(x)}{2(x - 1)} = 1 \Rightarrow f'(1) = 0,$

   $\lim_{x \to 1} \dfrac{f''(x)}{2} = 1 \Rightarrow f''(1) = 2$

   ∴ $f(x) = (x - 1)^2 + A(x - 1)^3 + B(x - 1)^4$

   $f'(x) = 2(x - 1) + 3A(x - 1)^2 + 4B(x - 1)^3$

   $f'(0) = -6 \Rightarrow 3A - 4B = -4$, and $f'(2) = 6$

   $\Rightarrow 3A + 4B = 4$

   $\Rightarrow A = 0$, $B = 1 \Rightarrow f(x) = (x - 1)^2 + (x - 1)^4$

   Subtangent $= \left|\dfrac{f(0)}{f'(0)}\right| = \dfrac{2}{6} = \dfrac{1}{3}.$

7. **(c) :** $\Sigma \tan A \tan B = 1 + \sec A \sec B \sec C$

   $= 1 - 8 = -7$

8. **(c) :** $\Sigma \tan A = \tan A \tan B \tan C = \dfrac{\sin A \sin B \sin C}{\cos A \cos B \cos C}$

   $= \dfrac{\Sigma \sin 2A}{4 \cos A \cos B \cos C} = -\sqrt{7}$

   $\Sigma \tan^2 A + 2\Sigma \tan A \tan B = 7$

   $\Sigma \tan^2 A = 21$, $\Sigma \sec^2 A = 24$.

9. $f(x) = \dfrac{6(ax^2 + bx + 1)}{(cx^2 + dx + 1)}$

   $f(2) = 3$, $f'(2) = 0$, $f(-2) = 4$, $f'(-2) = 0$

   Determining, $a = \dfrac{1}{4}$, $b = -3$, $c = \dfrac{1}{4}$, $d = -5$

   $f(x) = \dfrac{6(x^2 - 12x + 4)}{(x^2 - 20x + 4)}$, $f(1) = \dfrac{14}{5}$

   $m = 14$, $n = 5$, $m - n = 9$

10. **(a) → (t); (b) → (p); (c) → (q)**

    (a) coeff. of x^{10} in $(x + x^2 + ... + x^6)^4$

    = coeff. of x^6 in $(1 - x)^6 (1 - x)^{-4} = \binom{9}{3} - 4 = 80$

    ∴ Probability $= \dfrac{80}{6^4} = \dfrac{5}{81}$

    (b) Probability $= \dfrac{\binom{10 - 3 + 1}{3}}{\binom{10}{3}} = \dfrac{\binom{8}{3}}{\binom{10}{3}} = \dfrac{7}{15}$

    (c) Number of triangles not having common side with the octagon is $\dfrac{8}{6}(8 - 4)(8 - 5) = 16$

    ∴ Probability $= \dfrac{16}{^8C_3} = \dfrac{2}{7}$

■■

ACE

YOUR WAY **CBSE XI** | RELATIONS AND FUNCTIONS

HIGHLIGHTS

CARTESIAN PRODUCT OF SETS

Cartesian product of two sets A and B is denoted and defined as,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

- Cartesian product of two sets is not commutative.

Note :

- $(a, b) = (p, q) \Leftrightarrow a = p$ and $b = q$
- If $n(A) = p$, $n(B) = q$, then $n(A \times B) = pq$

RELATIONS

Any subset of $A \times B$ is a relation from A to B, where A and B are two non-empty sets.

So, R is a relation from A to B

$$\Leftrightarrow R \subseteq A \times B \Leftrightarrow R \subseteq \{(a, b) : a \in A, b \in B\}$$

Domain, Range and Codomain of a relation

Let R be a relation from A to B, then

- The set of all those elements $a \in A$ s.t. $(a, b) \in R$ for $b \in B$ is called domain of R.

Thus, Dom $(R) = \{a \in A : (a, b) \in R \text{ for } b \in B\}$

- The set of all those elements $b \in B$ s.t. $(a, b) \in R$ for $a \in A$ is called range of R.

Thus, Range of $R = \{b \in B : (a, b) \in R \text{ for } a \in A\}$

- The whole set B is called the co-domain of R.

Note : Range $\subseteq$ Co-domain.

FUNCTIONS

A relation f from set A to set B is said to be a function if

(i) Each element of A should have image in B.

(ii) No element of A should have more than one image in B.

Domain, Range and Co-domain of a function

Let $f : A \rightarrow B$ be a function, then

- Set A is the domain of f.
- The set of images of all elements of A is the range of f.
- Set B is the co-domain of f.

TYPES OF FUNCTIONS

Name of Function	Definition	Domain	Range	Graph
1. Identity Function	The function $f : R \rightarrow R$ defined by $f(x) = x \; \forall \; x \in R$	R	R	
2. Constant Function	The function $f : R \rightarrow R$ defined by $f(x) = c \; \forall \; x \in R$	R	$\{c\}$	

3. Polynomial Function	The function $f : R \to R$ defined by $f(x) = p_0 + p_1 x + p_2 x^2 + \ldots\ldots + p_n x^n$, where $n \in N$ and $p_0, p_1, p_2, \ldots, p_n \in R$ $\forall\ x \in R$			
4. Rational Function	The function f defined by $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions, $Q(x) \neq 0$			
5. Modulus Function	The function $f : R \to R$ defined by $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \quad \forall\ x \in R$	R	$[0, \infty)$	
6. Signum Function	The function $f : R \to R$ defined by $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \\ 0, & x = 0 \end{cases}$	R	$\{-1, 0, 1\}$	
7. Greatest Integer Function	The function $f : R \to R$ defined by $f(x) = [x] = \begin{cases} x, & x \in Z \\ \text{integer just less than } x, & x \notin Z \end{cases}$	R	Z	
8. Linear Function	The function $f : R \to R$ defined by $f(x) = mx + c$, $x \in R$ where, m and c are constants	R	R	

ALGEBRA OF REAL FUNCTIONS

Let $f : X \to R$ and $g : X \to R$ be two real functions, then

(i) **Addition :** $(f + g)(x) = f(x) + g(x) \ \forall\ x \in X$

(ii) **Subtraction :** $(f - g)(x) = f(x) - g(x) \ \forall\ x \in X$

(iii) **Multiplication :** $(fg)(x) = f(x) \cdot g(x) \ \forall\ x \in X$

(iv) **Division :** $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, g(x) \neq 0, x \in X$

(v) **Multiplication by a scalar :** Let α be any scalar, then, $(\alpha f)(x) = \alpha f(x), \ \forall\ x \in X$

Very Short Answer Type

1. If $A = \{1, 3, 5, 6\}$ and $B = \{2, 4\}$, find $A \times B$ and $B \times A$.

2. Let R be the relation on the set N of natural numbers defined by $R = \{(a, b) : a + 3b = 12, a \in N, b \in N\}$. Find domain of R.

3. If A and B are two sets given in such a way that $A \times B$ consists of 6 elements. And if three elements of $A \times B$ are (1, 3), (2, 5), (3, 3), then what are its remaining elements ?

4. If $f(x + 1) = 3x + 5$, then find $f(x)$.

5. Let $A = \{x, y, z\}$ and $B = \{1, 2\}$. Find the number of relations from A into B.

Long Answer Type - I

6. If $A \subseteq B$, show that $A \times A \subseteq (A \times B) \cap (B \times A)$.

7. Let $f : R \to R$ be defined by $f(x) = x$ and $g : R \to R$ be defined by $g(x) = |x|$. Find

(i) $f + g$ (ii) $f - g$ (iii) $f \cdot g$

(iv) $\alpha f, \alpha \in R$ (v) $\dfrac{f}{g}$.

8. Find the domain and range of the function $f(x)$ given by $f(x) = \dfrac{x-2}{3-x}$.

9. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, ..., 65\}$. Let R be a relation from A to B defined by aRb iff a is cube root of b. Find R and its domain and range.

10. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 5, 9, 11, 15, 16\}$ and $f = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$. Are the following true?
(i) f is a relation from A to B
(ii) f is a function from A to B
Justify your answer in each case.

11. Let f be the subset of $Z \times Z$ defined by $f = \{(ab, a + b) : a, b \in Z\}$. Is f a function from Z to Z ? Justify your answer.

12. Let A be a non-empty set such that $A \times B = A \times C$. Show that $B = C$.

13. Determine domain and range of the following relation. $\{(x, y) : y = |x - 1|, x \in Z \text{ and } |x| \le 3\}$

14. If $f(x) = \dfrac{x-1}{x+1}$, $x \ne -1$, then show that $f(f(x)) = -\dfrac{1}{x}$, provided that $x \ne 0$.

15. Let a relation R_1 on the set R of all real numbers be defined as $(a, b) \in R_1 \Leftrightarrow 1 + ab > 0$ for all $a, b \in R$. Show that :
(i) $(a, a) \in R_1$ for all $a \in R$
(ii) $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ for all $a, b \in R$

Long Answer Type-II

16. Find the range of the function : $f(x) = \dfrac{x}{1 + x^2}$

17. Find the domain and range of the function $\left\{ \left(x, \dfrac{1}{1 - x^2} \right) : x \in R, x \ne \pm 1 \right\}$.

18. Find the domain of function $f(x) = \log_4 \{\log_5 (\log_3 (18x - x^2 - 77))\}$

19. If A and B are any two non-empty sets, then prove that $A \times B = B \times A \Leftrightarrow A = B$.

20. Let f and g be real functions defined by $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{4 - x^2}$. Then, find each of the following functions :

(i) $f + g$ (ii) $f - g$ (iii) fg

(iv) $\dfrac{f}{g}$ (v) ff (vi) gg

SOLUTIONS

1. We have, $A = \{1, 3, 5, 6\}$ and $B = \{2, 4\}$. Therefore,
$A \times B = \{(1, 2), (1, 4), (3, 2), (3, 4), (5, 2), (5, 4), (6, 2), (6, 4)\}$
and $B \times A = \{(2, 1), (2, 3), (2, 5), (2, 6), (4, 1), (4, 3), (4, 5), (4, 6)\}$

2. $R = \{(a, b) : a = 12 - 3b, a \in N, b \in N\}$
$= \{(9, 1), (6, 2), (3, 3)\}$
$\therefore$ Domain of $R = \{9, 6, 3\}$

3. Since $(1, 3), (2, 5), (3, 3) \in A \times B$, so clearly $1, 2, 3 \in A$ and $3, 5 \in B$
Given, $n(A \times B) = 6 \Rightarrow n(A) \cdot n(B) = 6$
But $1, 2, 3 \in A$ and $3, 5 \in B$ i.e., $n(A) = 3, n(B) = 2$
Hence, $A = \{1, 2, 3\}$ and $B = \{3, 5\}$
$\therefore A \times B = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)\}$
$\therefore$ Remaining elements of $A \times B$ are : $(1, 5), (2, 3)$ and $(3, 5)$.

4. Given, $f(x + 1) = 3x + 5$ ...(i)
Putting $x - 1$ in place of x, we get
$f(x - 1 + 1) = 3(x - 1) + 5 \Rightarrow f(x) = 3x + 2$

5. Given, $A = \{x, y, z\}$ and $B = \{1, 2\}$
$\therefore n(A) = 3$ and $n(B) = 2$
$\therefore n(A \times B) = n(A) \cdot n(B) = 3 \cdot 2 = 6$
Total number of relations from A into B
= number of subsets of $A \times B = 2^6 = 64$.

6. Let (a, b) be an arbitrary element of $A \times A$. Then
$(a, b) \in A \times A$
$\Rightarrow a \in A$ and $b \in A$
$\Rightarrow (a \in A, b \in A)$ and $(a \in A, b \in A)$
$\Rightarrow (a \in A, b \in B)$ and $(a \in B, b \in A)$
$\qquad [\because A \subseteq B \therefore a, b \in A \Rightarrow a, b \in B]$
$\Rightarrow (a, b) \in (A \times B)$ and $(a, b) \in (B \times A)$
$\Rightarrow (a, b) \in (A \times B) \cap (B \times A)$
$\therefore A \times A \subseteq (A \times B) \cap (B \times A)$
Hence, $A \subseteq B \Rightarrow A \times A \subseteq (A \times B) \cap (B \times A)$.

7. $f + g, f - g, f \cdot g, \alpha \cdot f$ are functions from R to R defined by

(i) $(f + g)(x) = f(x) + g(x) = x + |x| = \begin{cases} 0, & x < 0 \\ 2x, & x \ge 0 \end{cases}$

(ii) $(f - g)(x) = f(x) - g(x) = x - |x| = \begin{cases} 2x, & x < 0 \\ 0, & x \ge 0 \end{cases}$

(iii) $(f \cdot g)(x) = f(x) \cdot g(x) = x |x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

(iv) $(\alpha \cdot f)(x) = \alpha \cdot f(x) = \alpha x$

(v) $\dfrac{f}{g} : R - \{0\} \to R$ is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{|x|} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$

8. We have, $f(x) = \dfrac{x-2}{3-x}$

Domain of f : Clearly, $f(x)$ is defined for all x satisfying $3 - x \neq 0$ *i.e.*, $x \neq 3$

Hence, Domain $(f) = R - \{3\}$

Range of f: Let $y = f(x)$. Then,

$$y = \frac{x-2}{3-x} \Rightarrow 3y - xy = x - 2$$

$$\Rightarrow \quad x(y+1) = 3y + 2 \Rightarrow x = \frac{3y+2}{y+1}$$

Clearly, x assumes real values for all y except $y + 1 = 0$ *i.e.*, $y = -1$

Hence, Range $(f) = R - \{-1\}$.

9. Here R is the relation 'is cube root of'.

So, we write $R = \{(a, b) : a = \sqrt[3]{b}, a \in A, b \in B\}$

Now, $1 = \sqrt[3]{1},\ 2 = \sqrt[3]{8},\ 3 = \sqrt[3]{27},\ 4 = \sqrt[3]{64},\ 5 = \sqrt[3]{125}$

Since, 1, 8, 27, 64 are in B but 125 is not in B.

$\therefore \quad R = \{(1, 1), (2, 8), (3, 27), (4, 64)\}$

Domain of $R = \{1, 2, 3, 4\}$

Range of $R = \{1, 8, 27, 64\}$.

10. (i) Here $A = \{1, 2, 3, 4\}$ and $B = \{1, 5, 9, 11, 15, 16\}$.

$\therefore \quad A \times B = \{(1, 1), (1, 5), (1, 9), (1, 11), (1, 15,),$
$\quad (1, 16), (2, 1), (2, 5), (2, 9), (2, 11), (2, 15),$
$\quad (2, 16), (3, 1), (3, 5), (3, 9), (3, 11), (3, 15), (3, 16),$
$\quad (4, 1), (4, 5), (4, 9), (4, 11), (4, 15), (4, 16)\}$

$f = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$

Now $(1, 5), (2, 9), (3, 1), (4, 5), (2, 11) \in A \times B$

$\therefore \quad f$ is a relation from A to B.

(ii) Here $f(2) = 9$ and $f(2) = 11$

$\therefore \quad f$ is not a function from A to B.

11. Let $ab = 10$

Then, $a = 2, b = 5$ or $a = 1, b = 10$
$\quad\quad a = -2, b = -5$ or $a = -1, b = -10$

Now, $a = 2, b = 5 \Rightarrow (ab, a + b) = (10, 7)$

and $\quad a = 1, b = 10 \Rightarrow (ab, a + b) = (10, 11)$

Since, 10 has two images in f, therefore, f is not a function.

12. Let b be an arbitrary element of B. Then,

$\quad\quad (a, b) \in A \times B$ for all $a \in A$

$\Rightarrow \quad (a, b) \in A \times C$ for all $a \in A \quad [\because A \times B = A \times C]$

$\Rightarrow \quad b \in C$

Thus, $b \in B \Rightarrow b \in C$

$\therefore \quad B \subseteq C \quad\quad\quad\quad\quad\quad\quad\quad ...(i)$

Now, let c be an arbitrary element of C. Then,

$\quad\quad (a, c) \in A \times C$ for all $a \in A$

$\Rightarrow \quad (a, c) \in A \times B$ for all $a \in A \quad [\because A \times B = A \times C]$

$\Rightarrow \quad c \in B$

Thus, $c \in C \Rightarrow c \in B$

$\therefore \quad C \subseteq B \quad\quad\quad\quad\quad\quad\quad\quad ...(ii)$

From (i) and (ii), we get $B = C$.

13. Let $R = \{(x, y) : y = |x - 1|, x \in Z$ and $|x| \leq 3\}$

Then, $R = \{(x, y) : y = |x - 1|, x \in Z$ and $-3 \leq x \leq 3\}$
$\quad\quad = \{(x, y) : y = |x - 1|, x = -3, -2, -1, 0, 1, 2, 3\}$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad ...(i)$

Now, $y = |x - 1|$

$\therefore \quad x = -3, -2, -1, 0, 1, 2, 3 \Rightarrow y = 4, 3, 2, 1, 0, 1, 2$

Hence from (i), $R = \{(-3, 4), (-2, 3), (-1, 2), (0, 1),$
$(1, 0), (2, 1), (3, 2)\}$

$\therefore \quad$ Domain of $R = \{-3, -2, -1, 0, 1, 2, 3\}$

Range of $R = \{4, 3, 2, 1, 0\} = \{0, 1, 2, 3, 4\}$.

14. We have, $f(x) = \dfrac{x-1}{x+1}, x \neq -1$

$$\Rightarrow \quad f(f(x)) = f\left(\frac{x-1}{x+1}\right) = \frac{\dfrac{x-1}{x+1} - 1}{\dfrac{x-1}{x+1} + 1}$$

$$= \frac{x-1-x-1}{x-1+x+1} = \frac{-2}{2x} = -\frac{1}{x}$$

Since, $-\dfrac{1}{x}$ is meaningful for $x \neq 0$.

Hence, $f(f(x)) = -\dfrac{1}{x}$, provided that $x \neq 0$.

15. (i) For any $a \in R$, we have

$\quad\quad 1 + a^2 > 0 \Rightarrow (a, a) \in R_1$

Thus, $(a, a) \in R_1$ for all $a \in R$.

(ii) Let $(a, b) \in R_1$. Then,

$\quad\quad 1 + ab > 0 \Rightarrow 1 + ba > 0 \Rightarrow (b, a) \in R_1$

Thus, $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ for all $a, b \in R$.
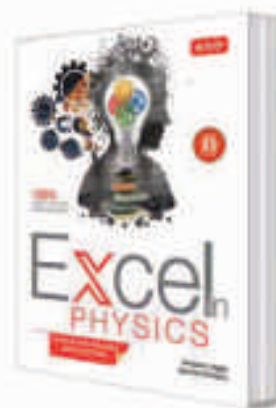
16. We have, $f(x) = \dfrac{x}{1+x^2}$

$\because \quad 1 + x^2 \neq 0$ for all $x \in R$

$\therefore \quad$ We observe that $f(x)$ takes real values for all $x \in R$. Hence, domain $(f) = R$
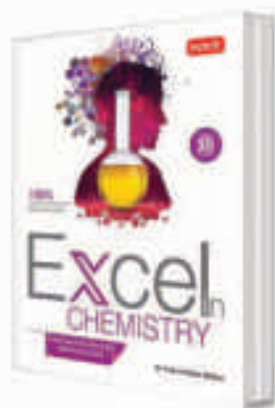
Let $y = f(x)$. Then,

$$y = \frac{x}{1+x^2} \Rightarrow x^2 y - x + y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1 - 4y^2}}{2y}$$
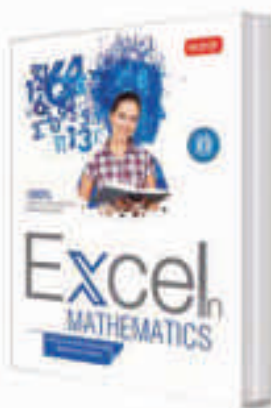
Clearly, x will assume real values, if
$$1 - 4y^2 \geq 0 \text{ and } y \neq 0$$
$$\Rightarrow 4y^2 - 1 \leq 0 \text{ and } y \neq 0 \Rightarrow y^2 - \frac{1}{4} \leq 0 \text{ and } y \neq 0$$
$$\Rightarrow \left(y - \frac{1}{2}\right)\left(y + \frac{1}{2}\right) \leq 0 \text{ and } y \neq 0$$
$$\Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2} \text{ and } y \neq 0 \Rightarrow y \in [-1/2, 1/2] - \{0\}$$
Also, $y = 0$ for $x = 0$
Hence, Range $(f) = [-1/2, 1/2]$

17. Let f be the given function, then
$$f(x) = \frac{1}{1 - x^2}$$
For $f(x)$ to be defined, $1 - x^2 \neq 0 \Rightarrow x^2 \neq 1 \Rightarrow x \neq \pm 1$
Hence, dom $(f) = R - \{-1, 1\} = \{x : x \in R \text{ and } x \neq \pm 1\}$
Range : Let $y = f(x)$, then $y = \frac{1}{1 - x^2}$
$$\Rightarrow 1 - x^2 = \frac{1}{y} \Rightarrow x^2 = 1 - \frac{1}{y} = \frac{y - 1}{y}$$
$$\Rightarrow x = \pm\sqrt{\frac{y - 1}{y}}$$
For x to be real, $\frac{y - 1}{y} \geq 0$
$$\Leftrightarrow -\infty < y < 0 \text{ or } 1 \leq y < \infty$$
Hence, Range $(f) = (-\infty, 0) \cup [1, \infty)$
$$= \{y : y < 0 \text{ or } y \geq 1\}.$$

18. We have, $f(x) = \log_4\{\log_5(\log_3(18x - x^2 - 77))\}$
Since $\log_a x$ is defined for all $x > 0$. Therefore, $f(x)$ is defined if
$$\log_5\{\log_3(18x - x^2 - 77)\} > 0 \text{ and } 18x - x^2 - 77 > 0$$
$$\Rightarrow \log_3(18x - x^2 - 77) > 5^0 \text{ and } x^2 - 18x + 77 < 0$$
$$\Rightarrow \log_3(18x - x^2 - 77) > 1 \text{ and } (x - 11)(x - 7) < 0$$
$$\Rightarrow 18x - x^2 - 77 > 3^1 \text{ and } 7 < x < 11$$
$$\Rightarrow 18x - x^2 - 80 > 0 \text{ and } 7 < x < 11$$
$$\Rightarrow x^2 - 18x + 80 < 0 \text{ and } 7 < x < 11$$
$$\Rightarrow (x - 10)(x - 8) < 0 \text{ and } 7 < x < 11$$
$$\Rightarrow 8 < x < 10 \text{ and } 7 < x < 11 \Rightarrow 8 < x < 10 \Rightarrow x \in (8, 10)$$
Hence, the domain of $f(x)$ is $(8, 10)$.

19. If part : Let $A = B$ ...(i)
To prove : $A \times B = B \times A$
Now, $A \times B = A \times A = B \times A$ [from (i)]
Only if part : Let $A \times B = B \times A$ ...(ii)
To prove : $A = B$
Let $a \in A$
Since B is non-empty $\therefore$ there exists $b \in B$
Now, $a \in A$ and $b \in B \Rightarrow (a, b) \in A \times B$

$\Rightarrow (a, b) \in B \times A$ [From (ii)]
$\Rightarrow a \in B$
Thus $a \in A \Rightarrow a \in B \therefore A \subseteq B$ ...(iii)
Let $b \in B$
Since A is non-empty $\therefore$ there exists $a \in A$
Now, $b \in B$ and $a \in A \Rightarrow (b, a) \in B \times A$
$\Rightarrow (b, a) \in A \times B$ [From (ii)]
$\Rightarrow b \in A$
Thus $b \in B \Rightarrow b \in A \therefore B \subseteq A$ ...(iv)
From (iii) and (iv), we have $A = B$.

20. We have, $f(x) = \sqrt{x + 2}$ and $g(x) = \sqrt{4 - x^2}$
Clearly, $f(x)$ is defined for all x satisfying
$$x + 2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x \in [-2, \infty)$$
$\therefore$ Domain $(f) = [-2, \infty)$
We observe that $g(x)$ is defined for all x satisfying
$$4 - x^2 \geq 0 \Rightarrow x^2 - 4 \leq 0 \Rightarrow (x - 2)(x + 2) \leq 0$$
$$\Rightarrow x \in [-2, 2]$$
$\therefore$ Domain $(g) = [-2, 2]$
Now, Domain $(f) \cap$ Domain (g)
$$= [-2, \infty) \cap [-2, 2] = [-2, 2]$$
(i) $f + g : [-2, 2] \to R$ is given by
$$(f + g)(x) = f(x) + g(x) = \sqrt{x + 2} + \sqrt{4 - x^2}$$
(ii) $f - g : [-2, 2] \to R$ is given by
$$(f - g)(x) = f(x) - g(x) = \sqrt{x + 2} - \sqrt{4 - x^2}$$
(iii) $fg : [-2, 2] \to R$ is given by
$$(fg)(x) = f(x)\, g(x) = \sqrt{x + 2} \times \sqrt{4 - x^2}$$
$$= \sqrt{(x + 2)^2 (2 - x)} = (x + 2)\sqrt{2 - x}$$
(iv) We have, $g(x) = \sqrt{4 - x^2}$
$\therefore g(x) = 0 \Rightarrow 4 - x^2 = 0 \Rightarrow x = \pm 2$
So, Domain $\left(\dfrac{f}{g}\right) = [-2, 2] - \{-2, 2\} = (-2, 2)$
$\therefore \dfrac{f}{g} : (-2, 2) \to R$ is given by
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{(x + 2)}}{\sqrt{4 - x^2}} = \frac{1}{\sqrt{2 - x}}$$
(v) Since domain $(f) = [-2, \infty)$. Therefore,
$$(ff)(x) = f(x)\, f(x) = [f(x)]^2 = \left(\sqrt{x + 2}\right)^2 = x + 2$$
for all $x \in [-2, \infty)$
(vi) Since domain $(g) = [-2, 2]$. Therefore,
$$(gg)(x) = g(x)\, g(x) = [g(x)]^2 = \left(\sqrt{4 - x^2}\right)^2$$
$$= 4 - x^2 \text{ for all } x \in [-2, 2].$$

OLYMPIAD CORNER

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1. Let a, b, c be positive real numbers. Show that there is a triangle with sides a, b, c if and only if there exist real numbers x, y, z such that
$$\frac{y}{z} + \frac{z}{y} = \frac{a}{x},\ \frac{z}{x} + \frac{x}{z} = \frac{b}{y},\ \frac{x}{y} + \frac{y}{x} = \frac{c}{z}.$$

2. Let x_1, x_2, x_3, x_4 be positive real numbers such that $x_1 x_2 x_3 x_4 = 1$. Prove that
$$\sum_{i=1}^{4} x_i^3 \geq \max\left\{\sum_{i=1}^{4} x_i,\ \sum_{i=1}^{4} \frac{1}{x_i}\right\}.$$

3. Let D be a point inside an acute triangle ABC such that $DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA = AB \cdot BC \cdot CA$.
Determine the geometric position of D.

4. Find all positive integers n that have exactly 16 positive integral divisors d_1, d_2, ..., d_{16} such that $1 = d_1 < d_2 < \cdots < d_{16} = n$, $d_6 = 18$ and $d_9 - d_8 = 17$.

5. Solve the following equation in natural numbers:
$x^2 + y^2 = 1997(x - y)$.

6. Find $x, y, z \in R$ satisfying
$$\frac{4\sqrt{x^2 + 1}}{x} = \frac{5\sqrt{y^2 + 1}}{y} = \frac{6\sqrt{z^2 + 1}}{z}\ \text{and}\ xyz = x + y + z.$$

7. Let ABC be a non-obtuse triangle such that $AB > AC$ and $\angle B = 45°$. Let O and I denote the circumcenter and incenter of ΔABC respectively. Suppose that $\sqrt{2}\,OI = AB - AC$. Determine all the possible values of $\sin\angle BAC$.

8. A triangle ABC has positive integer sides, $\angle A = 2\angle B$ and $\angle C > 90°$. Find the minimum length of the perimeter of ABC.

9. Let D, E, F be points on the sides BC, CA, AB, respectively, of triangle ABC. Let P, Q, R be the second intersections of AD, BE, CF, respectively, with the circumcircle of ABC. Show that
$$\frac{AD}{PD} + \frac{BE}{QE} + \frac{CF}{RF} \geq 9\ \text{and determine when equality}$$
occurs.

10. Find the number of polynomials of degree 5 with distinct coefficients from the set $\{1, 2, ..., 9\}$ that are divisible by $x^2 - x + 1$.

11. A triangle ABC has incentre I. Points X, Y are located on the line segment AB, AC respectively so that $BX \cdot AB = IB^2$ and $CY \cdot AC = IC^2$. Given that X, I, Y are collinear, find the possible values of the measure of angle A.

12. Prove that the polynomial $f(x) = x^4 + 26x^3 + 52x^2 + 78x + 1989$ cannot be expressed as product $f(x) = p(x)q(x)$ where $p(x), q(x)$ are both polynomials with integral coefficients and with degree not more than 3.

13. Determine all pairs (m, n) of positive integers for which $2^m + 3^n$ is a square.

14. Given a triangle ABC in a plane Σ find the set of all points P lying in the plane Σ such that the circumcircles of triangles ABP, BCP and CAP are congruent.

15. Determine all non-negative integral pairs (x, y) for which $(xy - 7)^2 = x^2 + y^2$.

16. Let $f : N \to N$ be a strictly increasing function such that $f(f(n)) = 3n$, for all natural numbers n. Find $f(2001)$.

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17. Show that there are no four consecutive binomial coefficients $\binom{n}{k}$, $\binom{n}{k+1}$, $\binom{n}{k+2}$, $\binom{n}{k+3}$ (n, k positive integers and $0 \le k + 3 \le n$) which are in arithmetic progression.

18. Find all functions $f : R \to R$ which satisfy the equation $f(x^2 + f(y)) = f(x)^2 + y$, for all $x, y \in R$.

19. Determine with proof, all the positive integers n for which
(i) n is not the square of any integer and
(ii) $[\sqrt{n}]^3$ divides n^2.
($[x]$ denotes the largest integer that is less than or equal to x.)

20. Let α be an irrational number. Then there are infinitely many integer pairs (h, k) where $k > 0$ such that $\left| \alpha - \dfrac{h}{k} \right| < \dfrac{1}{k^2}$.

21. Determine all triples (x, y, z) of positive integers which are solutions of $2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4 = 576$

22. Find all triples (x, y, z) of positive integers satisfying $2^x + 2^y + 2^z = 2336$.

23. Let ABC be a triangle and h_a be the altitude through A. Prove that $(b + c)^2 \ge a^2 + 4h_a^2$.
(As usual a, b, c denote the sides BC, CA, AB respectively.)

SOLUTIONS

1. If x, y, z exist, then some two of them share the same sign: say, x and y. Then $z = \dfrac{c}{\left(\dfrac{x}{y} + \dfrac{y}{x} \right)} > 0$. Thus
$a + b - c = \dfrac{2xy}{z}$, $b + c - a = \dfrac{2yz}{x}$, and $c + a - b = \dfrac{2zx}{y}$
are all positive, so a, b, c form a triangle.
Conversely, if there is a triangle with sides a, b, c then let $u = b + c - a$, $v = c + a - b$, $w = a + b - c$; by the triangle inequality, these are all positive. If there exist satisfactory x, y, z then from above
$u = \dfrac{2yz}{x}$, $v = \dfrac{2zx}{y}$, $w = \dfrac{2xy}{z}$. Solving these equations gives $x = \sqrt{\dfrac{vw}{2}}$, $y = \sqrt{\dfrac{wu}{2}}$, $z = \sqrt{\dfrac{uv}{2}}$, and these values indeed satisfy the equations.

2. Let $A = \sum x_i^3$ and $A_i = A - x_i^3$, so that $A = \dfrac{1}{3} \sum A_i$.
We claim that $A \ge \sum \dfrac{1}{x_i}$ and $A \ge \sum x_i$. From AM–GM, $\dfrac{1}{3} A_1 \ge \sqrt[3]{x_2^3 x_3^3 x_4^3} = \dfrac{1}{x_1}$.
Combining the analogous inequalities gives $A \ge \sum \dfrac{1}{x_i}$, as claimed. Also, by the power mean inequality,
$\dfrac{1}{4} A \ge \left(\dfrac{\sum x_i}{4} \right)^3 \ge \left(\dfrac{\sum x_i}{4} \right) \left(\dfrac{\sum x_i}{4} \right)^2 \ge \dfrac{\sum x_i}{4}$, since
$\Sigma x_i \ge 4$ by AM–GM. So $A \ge \Sigma x_i$, as claimed.

3. Let D be a point inside an acute triangle ABC. We have
$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA \ge AB \cdot BC \cdot CA$;
equality holds if and only if D is the orthocenter of ABC.

1st solution: Let E and F be points such that $BCDE$ and $BCAF$ are both parallelograms. Thus $EDAF$ is also a parallelogram. We have
$AF = ED = BC$, $EF = AD$, $EB = CD$, $BF = AC$.
Applying Ptolemy's theorem to quadrilaterals $ABEF$ and $AEBD$, we have
$AB \cdot AD + BC \cdot CD = AB \cdot EF + AF \cdot BE$
$\ge AE \cdot BF = AE \cdot AC$;
$BD \cdot AE + AD \cdot CD = BD \cdot AE + AD \cdot BE$
$\ge AB \cdot ED = AB \cdot BC$.
Now we have
$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA$
$= DB(AB \cdot AD + BC \cdot CD) + DC \cdot DA \cdot CA$
$\ge DB \cdot AE \cdot AC + DC \cdot DA \cdot CA$
$\ge AC(BD \cdot AE + AD \cdot CD)$
$\ge AC \cdot AB \cdot BC$.
Equality holds if and only if both $ABEF$ and $AEBD$ are cyclic, which implies that $AFEBD$ and $AFED$ are cyclic. Since $AFED$ is a parallelogram, $AFED$ is a rectangle and $AD \perp ED$. Since $BCDE$ is parallelogram, we have $ED \| BC$ and $AD \perp BC$. Since $AEBD$ is cyclic, $\angle ABE = \angle ADE$, which implies that $BE \perp AB$. Since $BCDE$ is a parallelogram, we have $CD \| EB$ and $CD \perp AB$. Thus D is the orthocenter of ABC.

2nd solution: Let D be the origin of the complex plane and let the complex coordinates of A, B, C be u, v, w, respectively. We rewrite as
$|uv(u - v)| + |vw(v - w)| + |wu(w - u)|$
$\ge |(u - v)(v - w)(w - u)|$. ...(1)

But it is easy to check that
$$uv(u - v) + uw(v - w) + wu(w - u)$$
$$= -(u - v)(v - w)(w - u), \qquad \ldots(2)$$
Now we only need to determine when the equality holds.

Let $z_1 = \dfrac{uv}{(u-w)(v-w)}$, $z_2 = \dfrac{vw}{(v-u)(w-u)}$,
$$z_3 = \dfrac{wu}{(w-v)(u-v)}.$$

We can rewrite (1) and (2) as
$$|z_1| + |z_2| + |z_3| \geq 1$$
$$z_1 + z_2 + z_3 = 1.$$
Equality holds if and only if z_1, z_2, z_3 are all positive real numbers.
Suppose that z_1, z_2, z_3 are all positive real numbers.

Since, $-\dfrac{z_1 z_2}{z_1} = \left(\dfrac{w}{u-v}\right)^2$, $-\dfrac{z_3 z_1}{z_2} = \left(\dfrac{u}{v-w}\right)^2$,

we know $\dfrac{u}{(v-w)}$ and $\dfrac{v}{(w-u)}$ are pure imaginary numbers; thus $AD \perp BC$ and $BD \perp AC$ and D is the orthocenter of ABC.

Suppose that D is the orthocenter of the triangle ABC. Since the triangle is acute, D is inside the triangle. Therefore there are some positive numbers r_1, r_2, r_3 such that
$$\dfrac{u}{v-w} = -r_1 i, \quad \dfrac{v}{w-u} = -r_2 i, \quad \dfrac{w}{u-v} = -r_3 i.$$
Thus z_1, z_2, z_3 are all positive real numbers.
From the above, we know that the equality holds if and only if D is the orthocenter of ABC.

4. Let integer $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ with $p_1, \ldots, p_m$ distinct primes. Then n has $(a_1 + 1)(a_2 + 1) \cdots (a_n + 1)$ divisors. Since $18 = 2 \cdot 3^2$, it has 6 factors: 1, 2, 3, 6, 9, 18. Since d has 16 divisors, we know that $d = 2 \cdot 3^3 \cdot p$ or $d = 2 \cdot 3^7$. If $b = 2 \cdot 3^7$, $d_8 = 54$, $d_9 = 81$ and $d_9 - d_8 \neq 17$. Thus $d = 2 \cdot 3^3 \cdot p$ for some prime $p > 18$. If $p < 27$, then $d_7 = p$, $d_8 = 27$, $d_9 = 2p = 27 + 17 = 44 \Rightarrow p = 22$, a contradiction. Thus $p > 27$. If $p < 54$, $d_7 = 27$, $d_8 = p$, $d_9 = 54 = d_8 + 17 \Rightarrow p = 37$. If $p > 54$, then $d_7 = 27$, $d_8 = 54$, $d_9 = d_8 + 17 = 71$. We obtain two solutions for the problem: $2 \cdot 3^3 \cdot 37 = 1998$ and $2 \cdot 3^3 \cdot 71 = 3834$.

5. The solutions are
$(x, y) = (170, 145)$ or $(1827, 145)$.
We have, $x^2 + y^2 = 1997(x - y)$
$2(x^2 + y^2) = 2 \times 1997(x - y)$

$x^2 + y^2 + (x^2 + y^2 - 2 \times 1997(x - y)) = 0$
$(x + y)^2 + ((x + y)^2 - 2 \times 1997(x - y)) = 0$
$(x + y)^2 + (1997 - x + y)^2 = 1997^2$.
Since x and y are positive integers, $0 < x + y < 1997$ and $0 < 1997 - x + y < 1997$. Thus the problem reduces to solving $a^2 + b^2 = 1997^2$ in positive integers. Since 1997 is a prime, $\gcd(a, b) = 1$. By Pythagorean substitution, there are positive integers $m > n$ such that $\gcd(m, n) = 1$ and
$1997 = m^2 + n^2$, $a = 2mn$, $b = m^2 - n^2$.
Since m^2, $n^2 \equiv 0, 1, -1 \pmod 5$ and $1997 \equiv 2 \pmod 5$, $m, n \equiv \pm 1 \pmod 5$. Since m^2, $n^2 \equiv 0, 1 \pmod 3$ and $1997 \equiv 2 \pmod 3$, $m > n$, $m, n \equiv \pm 1 \pmod 3$. Therefore $m, n \equiv 1, 4, 11, 14 \pmod{15}$. Since $m > n$, $\dfrac{1997}{2} \leq m^2 \leq 1997$. Thus we only need to consider $m = 34, 41, 44$. The only solution is $(m, n) = (34, 29)$. Thus
$$(a, b) = (1972, 315),$$
which leads to our final solutions.

6. Let $x = \tan\alpha$, $y = \tan\beta$, $z = \tan\gamma$,
$$\dfrac{-\pi}{2} < \alpha, \beta, \gamma < \dfrac{+\pi}{2}$$
$$\dfrac{4\sqrt{(\tan^2\alpha + 1)}}{\tan\alpha} = \dfrac{5\sqrt{(\tan^2\beta + 1)}}{\tan\beta} = \dfrac{6\sqrt{(\tan^2\gamma + 1)}}{\tan\gamma}$$
$$\Rightarrow \dfrac{4}{\sin\alpha} = \dfrac{5}{\sin\beta} = \dfrac{6}{\sin\gamma}$$
Again $\tan\alpha \tan\beta \tan\gamma = \tan\alpha + \tan\beta + \tan\gamma$
$\Rightarrow \tan\alpha(\tan\beta\tan\gamma - 1) = (\tan\beta + \tan\gamma)$
$$\Rightarrow -\tan\alpha = \dfrac{(\tan\beta + \tan\gamma)}{1 - \tan\beta\tan\gamma} = \tan(\beta + \gamma)$$
$\Rightarrow \tan(k\pi - \alpha) = \tan(\beta + \gamma) \Rightarrow \alpha + \beta + \gamma = k\pi$
Taking $k = 1$, we get $\alpha + \beta + \gamma = \pi$ which implies that there exists a Δ whose angles are α, β and γ and whose sides opposite to these angles are proportional to 4, 5 and 6 respectively.
Let the sides of such Δ be $4k$, $5k$ and $6k$.
$$s = \text{semi perimeter of the } \Delta = \dfrac{15k}{2}$$
$$\tan\dfrac{\alpha}{2} = \sqrt{\dfrac{(s - 5k)(s - 6k)}{s(s - 4k)}} = \sqrt{\dfrac{\frac{5k}{2} \times \frac{3k}{2}}{\frac{15}{2}k \times \frac{7}{2}k}} = \sqrt{\dfrac{1}{7}}$$
$$x = \tan\alpha = \dfrac{2t}{1 - t^2} = \dfrac{2\sqrt{\frac{1}{7}}}{1 - \frac{1}{7}} = \dfrac{\sqrt{7}}{3}$$

Similarly, $y = \tan\beta = \dfrac{5\sqrt{7}}{9}$ and $z = \tan\gamma = 3\sqrt{7}$

$$\left[\tan\dfrac{\beta}{2} = \sqrt{\dfrac{(s-4k)(s-6k)}{s(s-5k)}} \text{ and } \right.$$
$$\left. \tan\dfrac{\gamma}{2} = \sqrt{\dfrac{(s-4k)(s-5k)}{s(s-6k)}} \right]$$

7. **1ˢᵗ solution:** Let $a = BC$, $b = CA$, $c = AB$, $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$ and let R and r be the circumradius and inradius of ABC, respectively. Applying law of sines to ABC, we have
$a = 2R\sin\alpha$, $b = 2R\sin\beta$, $c = 2R\sin\gamma$

Since $\beta = 45°$, $\sin\beta = \dfrac{\sqrt{2}}{2}$, $\tan\left(\dfrac{\beta}{2}\right) = (\sqrt{2}-1)$

and $\sin\gamma = \sin(135° - \alpha) = \dfrac{\sqrt{2}(\sin\alpha + \cos\alpha)}{2}$ ...(1)

Thus, $r = \dfrac{(c+a-b)}{2}\tan\left(\dfrac{\beta}{2}\right)$

$= R(\sqrt{2}-1)(\sin\alpha + \sin\gamma - \sin\beta)$

From Euler's formula $OI^2 = R(R - 2r)$, we have
$OI^2 = R^2(1 - 2(\sin\alpha + \sin\gamma - \sin\beta)(\sqrt{2}-1))$ ...(2)
Since $\sqrt{2}OI = AB - AC$
$OI^2 = \left(c - 2(\sin\alpha + \sin\gamma - \sin\beta)(\sqrt{2}-1)\right)$ ...(3)
From (1) and (2), we obtain
$2(\sin\gamma - \sin\beta)^2$
$= \left(1 - 2(\sin\alpha + \sin\gamma - \sin\beta)(\sqrt{2}-1)\right)$
$\Leftrightarrow 1 - 2(\sin\gamma - \sin\beta)^2$
$= (\sin\alpha + \sin\gamma - \sin\beta)(\sqrt{2}-1)$
$\Leftrightarrow 1 - 2\sin^2\gamma + 2\sqrt{2}\sin\gamma - 1$
$= 2(\sin\alpha + \sin\gamma)(\sqrt{2}-1) - (2-\sqrt{2})$
$\Leftrightarrow -(\sin\alpha + \cos\alpha) + 2(\sin\alpha + \cos\alpha)$
$= (2\sqrt{2}-2)\sin\alpha + (2-\sqrt{2})(\sin\alpha + \cos\alpha) - (2-\sqrt{2})$
$\Leftrightarrow -1 - 2\sin\alpha\cos\alpha$
$= (\sqrt{2}-2)\sin\alpha - \sqrt{2}\cos\alpha - (2-\sqrt{2})$
$\Leftrightarrow 2\sin\alpha\cos\alpha - (2-\sqrt{2})\sin\alpha - \sqrt{2}\cos\alpha + (\sqrt{2}-1) = 0$
$\Leftrightarrow (\sqrt{2}\sin\alpha - 1)(\sqrt{2}\cos\alpha - \sqrt{2}+1) = 0$

Thus, $\sin\alpha = \dfrac{\sqrt{2}}{2}$ or $\alpha = 1 - \dfrac{\sqrt{2}}{2}$,

$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \dfrac{\sqrt{4\sqrt{2}-2}}{2}$.

2ⁿᵈ solution: Let I_c, I_a, I_b be the feet of perpendiculars from I to AB, BC, CA respectively. Let D be the foot of the perpendicular from O to BC. Thus OD is the perpendicular bisector of BC and $BD = CD$. From equal tangents, we have $AI_c = AI_b$, $BI_a = BI_c$, $CI_a = CI_b$. We have
$\sqrt{2}OI = c - b$
$= (AI_c + I_cB) - (AI_b + I_bC) = I_cB - I_bC = BI_a - I_aC$
Since $c > b$, D is on BI_a. We have $BI_a = BD + DI_a$, $I_aC = CD - DI_a$. So $\sqrt{2}OI = 2DI_a$, i.e., $OI = \sqrt{2}DI_a$.
Thus line OI and line DI_a form a 45° angle, which implies that either $OI \perp AB$ or $OI \parallel AB$.
(a) $OI \perp AB$. Then OI is the perpendicular bisector of AB. Thus $AC = BC$, $\alpha = \beta = 45°$ and
$\sin\alpha = \dfrac{\sqrt{2}}{2}$.
(b) $OI \parallel AB$. Let E be the foot of the perpendicular from O to AB. So $\angle AOE = \angle C = \gamma$,
$R\cos\angle AOE = R\cos\gamma = OE = II_c = r$.
Since, $r = 4R\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}$,

we have
$\cos\gamma = 4\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2} = 2\sin\dfrac{\beta}{2}\left(2\sin\dfrac{\alpha}{2}\sin\dfrac{\gamma}{2}\right)$
$= 2\sin\dfrac{\beta}{2}\left(-\cos\dfrac{\alpha+\gamma}{2} + \cos\dfrac{\alpha-\gamma}{2}\right)$
$= 2\sin\dfrac{\beta}{2}\left(-\sin\dfrac{\beta}{2} + \cos\dfrac{\alpha-\gamma}{2}\right)$
$= -2\sin^2\dfrac{\beta}{2} + 2\sin\dfrac{\beta}{2}\cos\dfrac{\alpha-\gamma}{2}$
$= \cos\beta - 1 + \sin\dfrac{\alpha+\beta-\gamma}{2} + \sin\dfrac{\beta+\gamma-\alpha}{2}$
$= \cos\beta - 1 + \sin(90° - \gamma) + \sin(90° - \alpha)$
$= \cos\beta - 1 + \cos\gamma + \cos\alpha$,

which implies that $\cos\alpha = 1 - \cos\beta = 1 - \dfrac{\sqrt{2}}{2}$ and

$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \dfrac{\sqrt{4\sqrt{2}-2}}{2}$.

8. **1ˢᵗ solution:** Let $BC = a$, $CA = b$, $AB = c$. We have $A = 2B$ and $C = 180° - 3B$. By the law of sines,
$$\dfrac{b}{\sin B} = \dfrac{a}{\sin A} = \dfrac{c}{\sin C}.$$
Since $\sin A = \sin 2B = 2\sin B\cos B$,
$\sin C = \sin 3B = 3\sin B - 4\sin^3 B$, we have
$a = 2b\cos B$, $c = b(3 - 4\sin^2 B) = b(4\cos^2 B - 1)$
and hence $a^2 = b(b + c)$. Since we are looking for

a triangle of smallest perimeter, we may assume that $\gcd(a, b, c) = 1$. In fact, $\gcd(b, c) = 1$, since any common factor of b and c would be a factor of a as well. We notice that, since a perfect square a^2 is being expressed as the product of two relatively prime integers b and c, it must be the case that both b and $b + c$ are perfect squares. Thus, for some integers m and n, with $\gcd(m, n) = 1$, we have $b = m^2$, $b + c = n^2$, $a = mn$, $2\cos B = \dfrac{n}{m} = \dfrac{a}{b}$. Since $C > 90°$, we have $0 < B < 30°$ and

$$\sqrt{3} < 2\cos B = \frac{n}{m} < 2.$$

It is easy to check that $(m, n) = (4, 7)$ is the smallest pair that generates a triangle $(a, b, c) = (28, 16, 33)$ that meets all the conditions.

2nd solution: We use same notations as those in the first solution. Let the angle bisector of $\angle CAB$ meet BC at D. Since $\angle BAD = \angle ABD$, we let $AD = BD = x$. We have $\angle ACD = \angle B$, $\angle ACB = \angle ACD$, so triangles ABC and DAC are similar. We have

$$\frac{x}{c} = \frac{b}{a} = \frac{a-x}{b}$$

which leads to $ax = bc$, $b^2 = a^2 - ax$
$\Rightarrow a^2 = b(b + c)$, and the rest is the same.

9. Fix ABC and note that $\dfrac{AD}{PD} = \dfrac{d(A, BC)}{d(P, BC)}$, which has

a constant numerator and so is minimized when the denominator is maximized, which occurs when P is the midpoint of the arc BC; and analogously for Q and R. Hence it suffices to prove the result when rays AD, BE, CF are angle bisectors. We have

$$\angle PBD = \frac{\angle BAC}{2} = \angle PAB \text{ and so triangles } PBD, PAB$$

are similar and

$$\frac{PA}{PD} = \frac{PA}{PB} \cdot \frac{PB}{PD} = \left(\frac{PA}{PB}\right)^2 = \left(\frac{AB}{BD}\right)^2.$$

But using the angle bisector theorem, $\dfrac{AB}{BD} = \dfrac{(b+c)}{a}$

and likewise $\dfrac{BC}{CE} = \dfrac{(c+a)}{b}$, $\dfrac{CA}{AF} = \dfrac{(a+b)}{c}$; now either expanding, regrouping, and using AM-GM or, more elegantly, using RMS-AM and AM-GM, as shown below, gives

$$\sum \frac{PA}{PD} = \sum \left(\frac{AB}{BD}\right)^2 \geq \frac{1}{3}\left(\sum \frac{AB}{BD}\right)^2 = \frac{1}{3}\left(\sum \frac{b+c}{a}\right)^2 \geq 12$$

and subtracting 3 from both sides gives our result.

Equality requires that AD, BE, CF be angle bisectors and (because of the AM-GM step) that ABC be equilateral.

10. Let the 5th degree equation be $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$. The roots of $x^2 - x + 1$ are the non-real roots of $x^3 + 1$, namely $e^{\pi i/3}$ and $e^{5\pi i/3}$. Therefore the 5th degree equation is divisible by $x^2 - x + 1$ iff
$ae^{5\pi i/3} + b^{4\pi i/3} + ce^{\pi i} + de^{2\pi i/3} + ee^{\pi i/3} + f = 0$.
In other words, so $i\sin60(-a - b + d + e) = 0$, or

$$a - d = e - b, \text{ and } \quad \frac{a}{2} - \frac{b}{2} - c - \frac{d}{2} + \frac{e}{2} + f = 0, \text{ on}$$

$e + 2f + a = b + 2c + d$ or (since $a - d = e - b$)
$a - d = c - f = e - b$. It follows that exactly $\dfrac{1}{12}$ of the polynomials will have coefficients $p + k$, q, $r + k$, p, $q + k$, r for $k > 0$ and $p \leq q \leq r$.

For a given k, there are $\dbinom{9-k}{3}$ values of p, q, r

such that $r + k \leq 9$. However, the coefficients must be distinct, so we must subtract those with 2 of p, q, r differing by k. There are $9 - 2k$ ways to select two numbers differing by k, and $7 - k$ ways to select the remaining number. However, we have counted those of the form x, $x + d$, $x + 2d$ twice, and there are $9 - 3k$ of these.

Therefore, for a given k, there are

$$\binom{9-k}{3} - (9-2k)(7-k) + 9 - 3k \text{ polynomials.}$$

Adding, we have $(1 + 4 + 10 + 20 + 35 + 56) - (42 + 25 + 12 + 3) + (3 + 6) = 53$ polynomials of the prescribed form, and $53 \cdot 12 = 636$ polynomials in total.

11. From the relation $BI^2 = BX \cdot BA$ we see that BI is a tangent to the circle passing through A, X, I at I.

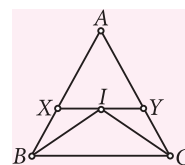Hence, $\angle BIX = \angle BAI = \dfrac{A}{2}$ ...(1)

[Alternatively, one observes that in triangles BIX and

BAI, $\angle IBX$ is common and $\dfrac{BI}{BX} = \dfrac{BA}{BI}$. Consequently

the two triangles are similar, implying (1).]
Similarly, from the relation $CI^2 = CY \cdot CA$ we obtain

$\angle CIY = \angle CAI = \dfrac{A}{2}$ ...(2)

It is known that

$\angle BIC = 90° + \dfrac{A}{2}$ ...(3)

From (1), (2), (3) and the fact that X, I, Y are collinear, we obtain

$$\frac{A}{2} + \frac{A}{2} + \left(90° + \frac{A}{2}\right) = 180°$$

On solving, we get $A = 60°$.

12. Assume, if possible,
$f(x) = (x + a)(x^3 + ax^2 + bx + c)$
Comparing the coefficients of like powers of x, we get, $a + b = 26$, $ab + c = 52$, $ac + d = 78$, $ad = 1989$.
But $1989 = 3^2 \cdot 13 \cdot 17$. Thus 13 divides ad and hence 13 divides a or d but not both. If 13 divides a then 13 divides $d = 78 - ac$ which is not possible. Suppose 13 divides d. Then 13 divides ac. But since 13 does not divide a, 13 divides c which implies 13 divides $ab = 52 - c$ and so b is divisible by 13 which in turn implies 13 divides $a = 26 - b$, a contradiction. Therefore $f(x)$ has no linear factors.
If $f(x) = (x^2 + ax + b)(x^2 + cx + d)$, then again,
$a = c = 26$, $b + ac + d = 52$, $ad + bc = 78$, $bd = 1989$.
Since $1989 = 3^2 \cdot 13 \cdot 17$, 13 divides bd. This implies that 13 divides b or d but not both. If 13 divides b, then 13 divides ad (= $78 - bc$) and hence 13 divides a. But then 13 divides d (= $52 - b - ac$), a contraction. Similar argument shows that 13 divides d is also not possible. We conclude that $f(x)$ cannot be written as a product of two polynomials with integral coefficients, each of degree < 4.

13. Suppose $2^m + 3^n = a^2$. Since any square number will leave remainder 0 or 1 when divided by 3 we get that m is an even number (as any odd power of 2 leaves remainder 2 when divided by 3). Similarly, using the fact that any square number is either divisible by 4 or will leave remainder 1 when divided by 4 we find that n is also an even number. Put $m = 2r$ and $n = 2s$. We have
$2^{2r} = a^2 - 3^{2s} = (a - 3^s)(a + 3^s)$
Hence, $(a - 3^s) = 2^i$ and $(a + 3^s) = 2^{2r - i}$.
We would then have $2 \cdot 3^s = 2^i(2^{2r - 2i} - 1)$, which implies that $i = 1$. Thus $a - 3^s = 2$ and $a + 3^s = (2^{2r - 2i} - 1)$ i.e., $3^s = 2^{2r - 2} - 1$. Suppose $s > 1$. Then $r \geq 3$. But then the above equation is impossible since when divided by 8, the left hand side 3^s would leave remainder 1 or 3 while the right hand side would leave the remainder 7. Thus $s = 1$ is the only possibility; when $s = 1$, that is $n = 2$, we have the solution $2^4 + 4^n = 25$. Thus $(m, n) = (4, 2)$ is the only solution.

14. **1st Solution:** We shall show that the locus of all such points is the union of the circumcircle and the orthocentre of the triangle ABC.
Let P be any point in the cone determined by two sides, (say) BA and BC. Using the sine rule in the triangles PAC and PBC, we get
$\angle CAP = \alpha$ or $180° - \alpha$
Similarly, using the triangles CAP and BAP, we also get
$\angle ACP = \beta$ or $180° - \beta$
Consider the case $\angle CAP = \alpha$ and $\angle ACP = 180° - \beta$
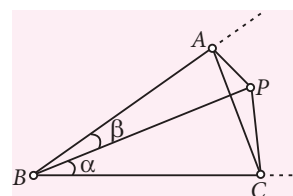
Here we get, $\angle APC = 180° - (\alpha + 180° - \beta) = \beta - \alpha$
Again the triangles BPC and BPA give
$\angle BAP = \angle BCP$ or $\angle BAP = 180° - \angle BCP$
If $\angle BAP = \angle BCP = \gamma$, then the sum of the angles of the quadrilateral is equal to $2\beta + 2\gamma$. This implies that $\beta + \gamma = 180°$. Since β and γ are angles of a triangle, this is impossible. If $\angle BAP = 180° - \angle BCP = 180° - \gamma$, then we get $-2\beta + 360° = 180°$. Hence $\beta = 90°$. This forces that $\angle PCA = 90°$ and AP is a diameter of the circle through A, B, C and P, i.e., P is on the circumcircle of $\triangle ABC$. Similarly, we can dispose off the case $\angle CAP = 180° - \alpha$, $\angle ACP = \beta$. Finally consider the case, $\angle CAP = 180° - \alpha$ and $\angle ACP = 180° - \beta$. Considering the triangle ACP, we see that $\angle APC = 180° - \angle ABC$
Similarly, the case $\angle CAP = \alpha$, $\angle ACP = \beta$ gives that $\angle APC$ and $\angle ABC$ are supplementary angles. Thus, A, B, C and P are concyclic.
On the other hand, suppose P is in the cone determined by the lines, say, CB and AB extended. Since
$\angle PBC + \angle PAC = \angle PBA + \angle PCA = 180°$,
it follows that $\angle ABC$ and $\angle APC$ are supplementary angles. Thus, triangles ABC and APC, and hence triangles ABC and BPC, have the same circumradii.
Now sine rule gives
$\angle CPB = \beta$ or $180° - \beta$, $\angle APB = \gamma$ or $180° - \gamma$
Also, if $\angle BAP = \alpha$, then $\angle BCP = \alpha$ or $180° - \alpha$.
Consider the cases.
Then
$\angle APC = \beta + 180° - \gamma$, $\angle PAC + \angle PCA = \beta + \gamma + 2\alpha$ and hence $\beta + \gamma + 2\alpha = \gamma - \beta$ or $\alpha + \beta = 0$ which is impossible. If $\angle BCP = 180° - \alpha$, then we have $\angle APC = \beta + 180° - \gamma$, $\angle PAC + \angle PCA = \beta + \gamma + 180°$

which is impossible. Similarly we can dispose off the cases
$\angle CPB = 180° - \beta$, $\angle APB = \gamma$, $\angle BCP = \alpha$ or $180° - \alpha$.
Finally if
$\angle CPB = \beta$, $\angle APB = \gamma$, $\angle BCP = 180° - \alpha$,
then again we get
$\angle APC = \beta + \gamma$, $\angle PAC + \angle PCA = 180° + \beta + \gamma$
This forces $2(\beta + \gamma) = 0$ which is impossible. We conclude that the only possibility is
$\angle APB = \gamma$, $\angle CPB = \beta$ and $\angle BCP = \alpha$
In this case, we get
$\angle APC = \beta + \gamma$, $\angle PAC + \angle PCA = 2\alpha + \beta + \gamma$
This gives us, $\alpha = 90° - (\beta + \gamma)$
Thus $\beta + \alpha = 90° - \gamma$ and $\alpha + \gamma = 90° - \beta$. These imply that AP is perpendicular to CB and CP is perpendicular to AB. Hence P is the orthocentre. Similarly we can consider other regions determined by BA and CA or BC and AC.

Finally if P is a point inside the triangle, we can show that P is the orthocentre of the triangle ABC in the similar way.

Thus if P is any point satisfying the hypothesis, then either P is the orthocentre of the triangle ABC or P must be on the circumcircle of the triangle ABC.

2nd Solution: We need to know the following facts about three equal circles intersecting in a common point. If three congruent (that is, equal) circles C_1, C_2, C_3 have a common point P and A, B, C are the other three points of intersections, then
(a) the circumcircle of triangle ABC has the same radius as the three circles; and
(b) the point P is the orthocentre of triangle ABC.
A brief proof of (a) and (b) follows:
Let X, Y, Z be the centres of the circles C_1, C_2, C_3 respectively. Complete the quadrilaterals $PXBZ$ and $PXCY$, join AP and ZY. Observe that $PXBZ$ and $PXCY$ are rhombuses and so ZB is parallel and equal to YC. Hence so are BC and ZY. Since AP is perpendicular to ZY, AP is perpendicular to BC. Similarly BP and CP are perpendicular to CA and AB respectively. Hence P is the orthocentre of triangle ABC. This proves (b).
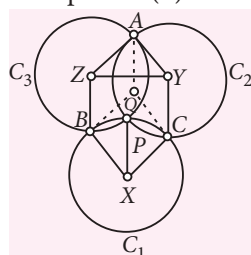
To prove (a), complete the parallelogram $AYCQ$, which is in fact a rhombus. So $AQ = CQ$. It is easily seen that $AZBQ$ is also a rhombus. So $AQ = BQ$. Thus Q is circumcentre of triangle ABC and its radius ($= AQ = CY$) is the same as that of each of the three circles. Note that we can have a configuration of three equal circles such that P falls outside triangle ABC, but statements (a) and (b) are still true.

Coming to the problem, let (XYZ) denote the circle through any three non collinear points X, Y, Z. It is given that three equal circles pass through P. Hence by (a) above, the four circles (PAB), (PBC), (PCA) and (ABC) are congruent to one another. Observe that either the three circles (PAB), (PBC), (PCA) coincide [and hence coincide with (ABC)] or the three circles are all distinct passing through the point P. Thus either P is on the circumcircle of ABC or P is the orthocentre of ABC.

15. We have the obvious solution (7, 0) and (0, 7). So suppose $x \neq 0$ and $y \neq 0$. We have
$(xy - 7)^2 = x^2 + y^2$
or $(xy)^2 - 14xy + 49 = x^2 + y^2$
or $(xy)^2 - 12xy + 36 + 13 = x^2 + y^2 + 2xy$
or $(xy - 6)^2 + 13 = (x + y)^2$
or $13 = [(x + y) + (xy - 6)] [(x + y) - (xy - 6)]$
Since 13 is prime number the only possible factors are ± 1 and ± 13, *i.e.*,
(i) $(x + y) + (xy - 9) = 13$
and $(x + y) - (xy - 6) = 1$
                    or
(ii) $(x + y) - (xy - 6) = -13$
and $(x + y) + (xy - 6) = -1$.
When solved, these alternatives give the solutions (3, 4) and (4, 3). Thus, (7, 0), (0, 7), (3, 4) and (4, 3) are all the solutions (in non-negative integers) of $(xy - 7)^2 = x^2 + y^2$.

16. An immediate consequence of the given relation is that f is an one-one function on N. We have form the given relation, $f(3n) = f(f(f(n))) = 3f(n)$, $\forall n \in N$.
It follows that $f(3) = 3f(1)$. If $f(1) = 1$, then we obtain
    $3 = 3 \cdot 1 = f(f(1)) = f(1) = 1$,
which is absurd. It follows that $f(1) > 1$ and hence
    $3 = f(f(1)) > f(1) > 1$,
where we have used the fact that f is strictly increasing. The only possibility, therefore, is $f(1) = 2$.

This in turn implies that $f(2) = f(f(1)) = 3$. Since $2001 = 3 \cdot 667$, it is sufficient to compute $f(667)$.

We shall get an expression for $f(k)$, for any k in N. We observe that

$$f(3) = 3f(1) = 6, \ f(6) = f(3 \cdot 2) = 3f(2) = 9$$

Since f is strictly increasing, we also note that,

$$6 = f(3) < f(4) < f(5) < f(6) = 9$$

This completely determines the intermediate values of f; $f(4) = 7, f(5) = 8$.

These values in turn give, $f(7) = f(f(4)) = 3 \cdot 4 = 12$, $f(8) = f(f(5)) = 15$, $f(9) = f(f(6)) = 18$. Now using $f(7) = 12$, we obtain

$f(12) = f(f(7)) = 3 \cdot 7 = 21$. The values $f(9) = 18$ and $f(12) = 21$ together with the fact that f is strictly increasing now determines $f(10) = 19$ and $f(11) = 20$.

Suppose for some k, we have $f(k) = n$ and $f(k + 1) = n + 1$. Then we see that

$f(n) = f(f(k)) = 3k$, $f(n + 1) = f(f(k + 1)) = 3k + 3$. If $f(k) = n$ and $f(k + 1) = n + 3$, then $f(n) = 3k$, $f(n+3) = 3k+3$, and these values fix $f(n+1)$ and $f(n+2)$; $f(n + 1) = 3k + 1$, $f(n + 2) = 3k + 2$. Let n be such that $3^m \le n < 2 \cdot 3^m$, for some m. In this case

$f(3^m) = 3^m f(1) = 2 \cdot 3^m$, $f(2 \cdot 3^m) = f(f(3^m))$
$= 3 \cdot 3^m = 3^{m+1}$.

Note that, because f is strictly increasing

$$2 \cdot 3^m = f(3^m) < f(3^m + 1) < \dots$$
$$< f(3^m + 3^m - 1) < f(2 \cdot 3^m)3^{m+1},$$

and hence we get

$f(3^m + j) = 2 \cdot 3^m + j$, for $0 \le j \le 3^m$.

Thus $f(n) = n + 3^m$ for all n such that $3^m \le n \le 2 \cdot 3^m$. If $2 \cdot 3^m \le n \le 3^{m+1}$, then $n = 2 \cdot 3^m + j$, where $0 \le j \le 3^m$. Hence

$f(n) = f(2 \cdot 3^m + j) = f(f(3^m + j))$
$\quad = 3(3^m + j) = 3^{m+1} + 3j = 3n - 3^{m+1}$

We have thus obtained the following description of $f(n)$:

$$f(n) = \begin{cases} n + 3^m, & \text{for } 3^m \le n \le 2 \cdot 3^m \\ 3n - 3^{m+1}, & \text{for } 2 \cdot 3^m \le n \le 3^{m+1}. \end{cases}$$

Since $2001 = 3 \cdot 667$, we obtain $f(2001) = 3f(667)$. We observe that $3^5 = 243$, $2 \cdot 3^5 = 486$ and $3^6 = 729$. Thus 667 lies between $2 \cdot 3^5$ and 3^6. Using the description of f, we obtain

$f(667) = 3 \cdot 667 - 3^6 = 3(667 - 243) = 1272$.

Thus the required value is $f(2001) = 3(1272) = 3816$.

17. The binomial coefficients fill a triangular array known as **Pascal's Triangle** in which their computation is easily accomplished using the relation

$$\binom{r+1}{s+1} = \binom{r}{s} + \binom{r}{s+1}.$$

Suppose, $\binom{n}{k}$, $\binom{n}{k+1}$, $\binom{n}{k+2}$, $\binom{n}{k+3}$

are in arithmetic progression and consider the following portion of Pascal's triangle:

$$\binom{n}{k} \binom{n}{k+1} \binom{n}{k+2} \binom{n}{k+3}$$
$$\binom{n+1}{k+1} \binom{n+1}{k+2} \binom{n+1}{k+3}$$
$$\binom{n+2}{k+2} \binom{n+2}{k+3}$$

Following our assumption, the entries can be filled into produce the following array:

$a \quad a + d \quad a + 2d \quad a + 3d$
$2a + d \quad 2a + 3d \quad 2a + 5d$
$4a + 4d \quad 4a + 8d$

Thus

$$\frac{\binom{n+2}{k+2}}{\binom{n}{k+1}} = \frac{\binom{n+2}{k+3}}{\binom{n}{k+1}} = 4.$$

Suppose the equality on the left holds. Then

$$\frac{1}{(k+2)(n-k)} = \frac{1}{(k+3)(n-k-1)},$$

which yields $n = 2k + 3$. But then the equality on the right doesn't hold:

$$\frac{\binom{2k+5}{k+2}}{\binom{2k+3}{k+1}} = \frac{2(2k+5)}{k+3} = 4 - \frac{2}{k+3} \ne 4$$

Thus our assumption that

$\binom{n}{k}$, $\binom{n}{k+1}$, $\binom{n}{k+2}$, $\binom{n}{k+3}$

are in arithmetic progression has produced a contradiction.

18. **1st Solution:**

$$f(x^2 + f(y)) = f(x)^2 + y \qquad \dots(1)$$

Taking $x = 0$ in (1) and putting $f(0) = s$, we get

$$f(f(y)) = s^2 + y \text{ for all } y \in R. \qquad \dots(2)$$

Similarly taking $y = 0$ in (1), we obtain

$\quad f(x^2 + s) = f(x)^2$, for all $x \in R \qquad$ ...(3)

Setting $x = 0$ in (3) leads to the relation,

$\quad f(s) = s^2 \qquad\qquad$ ...(4)

Addition of (3) and (4) gives

$s^2 + f(x^2 + s) = f(x^2) + f(s)$

This implies that

$f(s^2 + f(x^2 + s)) = f(f(x^2) + f(s))$

We reduce the above relation to

$f(s)^2 + x^2 + s = (f(f(x)))^2 + s$

If we now use (2) and (4), we see that

$s^2 + x^2 + s = (s^2 + x)^2 + s$

This simplifies to $2s^2x = 0$, valid for all $x \in R$, which is possible only if $s = 0$. Using this fact in (2) and (3), we get

$\quad f(f(y)) = y$, for all $y \in R$

and $f(x^2) = f(x)^2$, for all $x \in R$. $\qquad$ ...(6)

We observe that (6) implies $f(x) \geq 0$ if $x \geq 0$. If $f(x) = 0$ for some x, then

$f(x^2) = f(x^2 + f(x)) = f(x)^2 + x = x$,

so that $x = f(x^2) = f(x)^2 = 0$. It follows that $f(x) > 0$ if $x > 0$.

Replacing x by $f(x)$ in (1), we get

$\quad f(f(x)^2 + f(y)) = (f(f(x)))^2 + y = x^2 + y$

This in turn gives

$\quad f(x^2 + y) = f(x)^2 + f(y) = f(x^2) + f(y)$

Thus we get a restricted form of additivity;

$f(z + y) = f(z) + f(y)$ for all $z \geq 0$ and all real y.

Suppose we take two real numbers x, y such that $x > y$. Then $x - y > 0$ and hence

$\quad f(x) = f(x - y + y) = f(x - y) + f(y) > f(y)$;

we have used the fact that $x - y > 0$ and the restricted additivity which we have property of f that it is strictly increasing on R. This is enough to fix the values of f. If $f(x) > x$ for some x, then the strictly increasing nature of f give $f(f(x)) > f(x)$. But f is involutive; i.e., $f(f(x)) = x$ for all $x \in R$. We thus arrive at $x > f(x)$ contradicting what we have started with. Similarly, we can easily check that $f(x) < x$ is also not possible. The only left-out option is $f(x) = x$ for all $x \in R$. It is easy to verify that this function satisfies the given equation.

2ⁿᵈ Solution:

We see from the given equation that $f(f(y)) = y + (f(0))^2$.

Suppose $f(y) < y$ for some y. Then we can find x such that $y - f(y) = x^2$. This leads to

$\quad f(y) = f(x^2 + f(y)) = y + f(x)^2$,

showing that $y \leq f(y)$. It follows that $y \leq f(y)$, for all $y \in R$.

Now choose $y_0 < -f(0)^2$ and consider $\alpha = f(y_0)$. We see that

$\alpha \leq f(\alpha) = f(f(y(0))) = y_0 + f(0)^2 < 0$

Thus $\alpha, f(\alpha)$ are both negative and $\alpha \leq f(\alpha)$. It follows that $f(\alpha)^2 \leq \alpha^2$. Take any $x \in R$, we observe that

$\alpha^2 + x \leq \alpha^2 + f(x) \leq f(\alpha^2 + f(x)) = x + f(\alpha)^2 \leq x + \alpha^2$.

Thus there is equality throughout and this gives $f(x) = x$.

19. Let $\left[\sqrt{n}\right] = k$. Then $k^2 < n < (k + 1)^2$. Also since k^3 divides n^2, we have that k^2 divides n^2 and hence k divides n. Thus the only possibilities for n are $n = k^2 + k$ and $n = k^2 + 2k$.

(i) Let $n = k^2 + k$. Then

$\quad k^3 \mid n^2 \Rightarrow k^3 \mid (k^2 + k)^2 = k^4 + 2k^3 + k^2$

$\Rightarrow k^3 \mid k^2 \Rightarrow k = 1$

i.e., $n = 2$.

(ii) Let $n = k^2 + 2k$. Then

$\quad k^3 \mid n^2 \Rightarrow k^3 \mid (k^2 + 2k)^2 = k^4 + 4k^3 + 4k^2$

which implies that $k^3 \mid 4k^2$ or $k \mid 4$. Therefore, $k = 1$, 2 or 4. When $k = 1, 2, 4$ we get the corresponding values 3, 8 and 24 for n. Thus $n = 2, 3, 8$ and 24 are all positive integers satisfying the given conditions.

20. Let Q be an arbitrary positive integer. We claim that there exist integers h and k, with $1 \leq k \leq Q$, such that

$$\left|\alpha - \frac{h}{k}\right| < \frac{1}{kQ}.$$

This implies that the desired infinite collection of rational numbers exists. Otherwise, there would be a positive number ε such that $\left|\alpha - \dfrac{h}{k}\right| > \varepsilon$ for all (h, k) satisfying. But then, by choosing $Q > \dfrac{1}{\varepsilon}$, yields

$$\left|\alpha - \frac{h}{k}\right| < \frac{1}{kQ} \leq \min\left(\frac{1}{k^2}, \frac{1}{Q}\right) \leq \min\left(\frac{1}{k^2}, \varepsilon\right),$$

a contradiction. Now we need to prove the claim, and here is where the Pigeonhole Principle is used. The "boxes" are the intervals

$$B_k = \left\{ x \,\middle|\, \frac{k-1}{Q} \leq x < \frac{k}{Q} \right\}, \quad k = 1, 2, \dots, Q$$

and the "objects" are the numbers $\{q\alpha\}(q = 0,1,2,...,Q)$, where $\{x\} = x - [x]$ denotes the fractional part of x. Since there are Q boxes and $Q + 1$ objects, some box must contain at least two objects. This and $Q + 1$ objects, some box must contain at least two objects. This implies

$$\left|\{q_1\alpha\} - \{q_2\alpha\}\right| < \frac{1}{Q} \text{ for some } 0 \le q_1 < q_2 \le Q.$$ Set $h = m_2 - m_1$ and $k = q_2 - q_1$ where $m_1 = [q_1\alpha]$, $m_2 = [q_2\alpha]$. This gives

$$\left|\alpha - \frac{h}{k}\right| < \frac{1}{kQ}$$

with $1 \le k \le Q$.

21. We have to find the positive integers x, y, z such that
$2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4 = 576$
Let
$E = 2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4$
$= 4x^2y^2 - [x^4 + y^4 - z^4 + 2x^2y^2 - 2y^2z^2 - 2z^2x^2]$
$= 4x^2y^2 - [x^2 + y^2 - z^2]^2$
$= (2xy)^2 - (x^2 + y^2 - z^2)^2$
$= (2xy + x^2 + y^2 - z^2)(2xy + z^2 - x^2 - y^2)$
$= [(x + y)^2 - z^2][z^2 - (x - y)^2]$
$= (x + y + z)(x + y - z)(z + x - y)(z - x + y)$
∵ The equation becomes
$(x + y + z)(x + y - z)(x - y + z)(-x + y + z) = 576$
x, y, z are positive integers. We find that $x + y + z$
$= (x + y - z) + 2z$
⇒ All the factors are of same parity.
⇒ All of them must be even. Let
$x + y + z = 2a$, $x + y - z = 2b$, $x - y + z = 2c$, $-x + y + z = 2d$
⇒ $abcd = 36$
Without loss of generality assume $x \ge y \ge z$
⇒ $a > b \ge c \ge d$.
We observe $a = b + c + d$.
$36 = 6 \times 3 \times 2 \times 1$. This factorization is unique.
⇒ $a = 6, b = 3, c = 2, d = 1$
$x = 5, y = 4, z = 3$
Exploiting the symmetry of the equation, x, y, z can be cyclically changed.

22. We have to find positive integers x, y, z such that
$2^x + 2^y + 2^z = 2336$.
Let us first express 2336 in powers of 2 if possible.
$2336 = 2^5 \times 73$
∴ $2^x + 2^y + 2^z = 2^5 \times 73$

$\Rightarrow \frac{2^x}{2^5} + \frac{2^y}{2^5} + \frac{2^z}{2^5} = 73 \Rightarrow 2^{x-5} + 2^{y-5} + 2^{z-5} = 73$

RHS is odd. LHS being powers of 2 is even
∴ One of the terms must be odd. This is possible only when 2^{x-5} or 2^{y-5} or 2^{z-5} is 1
Let $2^{x-5} = 1 \Rightarrow x - 5 = 0 \Rightarrow x = 5$
$\Rightarrow 1 + 2^{y-5} + 2^{z-5} = 73 \Rightarrow 2^{y-5} + 2^{z-5} = 72$
$2^{y-5} + 2^{z-5} = 8 \times 9 = 2^3 \times 9$
$\Rightarrow 2^{y-8} + 2^{z-8} = 9$
Again RHS is odd, LHS is even. Thus one of the two terms is odd.
Let $2^{y-8} = 1 \Rightarrow y - 8 = 0 \Rightarrow y = 8$
∴ $2^{z-8} = 8 = 2^3$
$\Rightarrow z - 8 = 3 \Rightarrow z = 11$
Since the equation is symmetric in x, y, z the solution set is
$(x, y, z) = (5, 8, 11), (8, 5, 11), (11, 5, 8)$
$(5, 11, 8), (8, 11, 5), (11, 8, 5)$

23. **1st Solution :** Draw a line l parallel to BC through A and reflect AC in this line to get AD. Let CD intersect l in P. Join BD
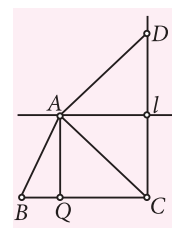
Observe that $CP = PD = AQ = h_a$, AQ being the altitude through A. We have

$$b + c = AC + AB = AD + AB \ge BD = \sqrt{CD^2 + CB^2}$$

$$= \sqrt{4h_a^2 + a^2}$$

which yields the result. Equality occurs if and only if B, A, D are collinear, *i.e.*, if and only if $AD = AB$ (as AP is parallel to BC and bisects DC) and this is equivalent to $AC = BC$.

2nd Solution : The given inequality is equivalent to

$$(b+c)^2 - a^2 \ge 4h_a^2 = \frac{16\Delta^2}{a^2},$$

Where Δ is the area of the triangle ABC. Using the identity $16\Delta^2 = [(b + c)^2 - a^2][a^2 - (b - c)^2]$
we see that the inequality to be proved is $a^2 - (b - c)^2 \le a^2$ (here we use $a < b + c$) which is true. Observe that equality holds if and only if $b = c$.
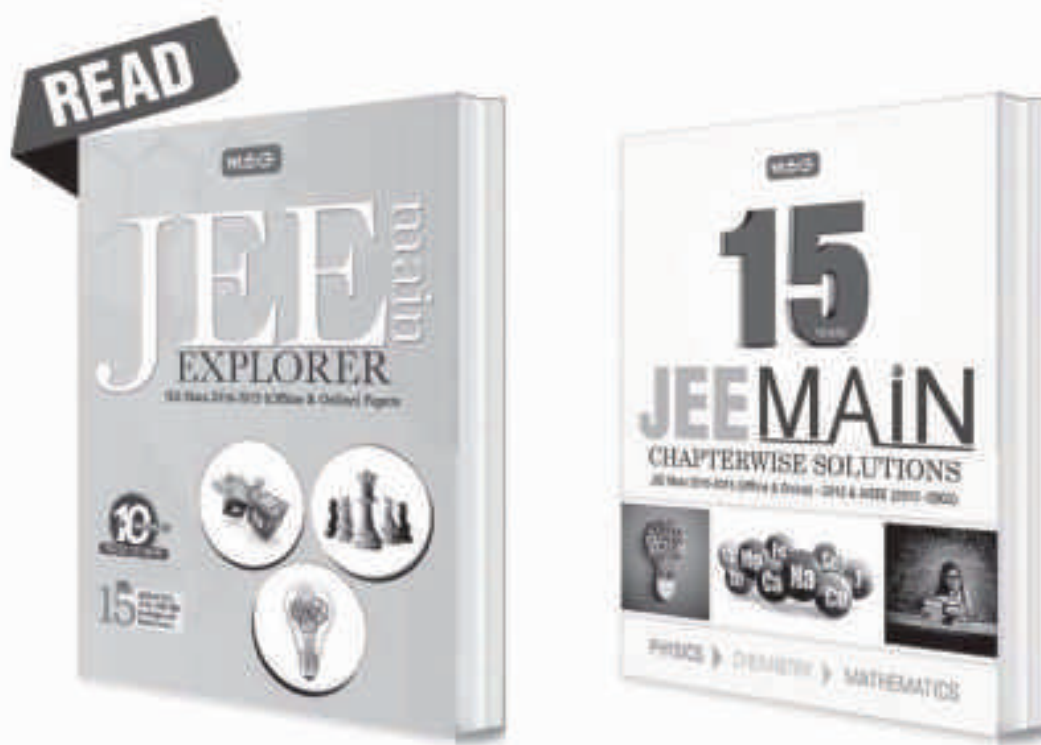
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ACE

YOUR WAY CBSE XII — INVERSE TRIGONOMETRIC FUNCTIONS

HIGHLIGHTS

Functions	Domain	Range (Principal Value Branch)	Graph
$\sin^{-1}x$	$[-1, 1]$	$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$	
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$	
$\tan^{-1}x$	$(-\infty, \infty)$	$\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$	
$\cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$	

$\operatorname{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$	
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\dfrac{\pi}{2}\right\}$	

Note : $(\sin x)^{-1} = \dfrac{1}{\sin x}$

Principal Value : The value of an inverse trigonometric function which lies in its principal value branch.

<div align="center">

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

</div>

1.
- $\sin^{-1}(\sin x) = x, \forall x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
- $\cos^{-1}(\cos x) = x, \forall x \in [0, \pi]$
- $\tan^{-1}(\tan x) = x, \forall x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
- $\cot^{-1}(\cot x) = x, \forall x \in (0, \pi)$
- $\sec^{-1}(\sec x) = x, \forall x \in [0, \pi] - \left\{\dfrac{\pi}{2}\right\}$
- $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x, \forall x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$

2.
- $\sin(\sin^{-1}x) = x, \forall x \in [-1, 1]$
- $\cos(\cos^{-1}x) = x, \forall x \in [-1, 1]$
- $\tan(\tan^{-1}x) = x, \forall x \in R$
- $\cot(\cot^{-1}x) = x, \forall x \in R$
- $\sec(\sec^{-1}x) = x, \forall x \in R - (-1, 1)$
- $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x, \forall x \in R - (-1, 1)$

3.
- $\sin^{-1}(-x) = -\sin^{-1}x, \forall x \in [-1, 1]$
- $\cos^{-1}(-x) = \pi - \cos^{-1}x, \forall x \in [-1, 1]$
- $\tan^{-1}(-x) = -\tan^{-1}x, \forall x \in R$
- $\cot^{-1}(-x) = \pi - \cot^{-1}x, \forall x \in R$
- $\sec^{-1}(-x) = \pi - \sec^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$
- $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$

4.
- $\sin^{-1}(1/x) = \operatorname{cosec}^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$
- $\cos^{-1}(1/x) = \sec^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$
- $\tan^{-1}(1/x) = \begin{cases} \cot^{-1} x & , \quad \text{for } x > 0 \\ -\pi + \cot^{-1} x & , \quad \text{for } x < 0 \end{cases}$

5.
- $\sin^{-1}x + \cos^{-1}x = \pi/2, \forall x \in [-1, 1]$
- $\tan^{-1}x + \cot^{-1}x = \pi/2, \forall x \in R$
- $\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2, \forall x \in (-\infty, -1] \cup [1, \infty)$

6.	
• $\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & ,\text{ if } x>0,\ y>0,\ xy<1 \\[2mm] \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & ,\text{ if } x>0,\ y>0 \text{ and } xy>1 \\[2mm] -\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{ if } x<0,\ y<0 \text{ and } xy>1 \end{cases}$	
• $\tan^{-1}x - \tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & ,\text{ if } x>0,\ y<0,\ xy>-1 \\[2mm] \pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & ,\text{ if } x>0,\ y<0 \text{ and } xy<-1 \\[2mm] -\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{ if } x<0,\ y>0 \text{ and } xy<-1 \end{cases}$	

7.	
• $\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} & ,\ \text{if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \\ & \text{or} \\ & \text{if } xy<0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}, & \text{if } 0<x,\ y\le 1 \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}, & \text{if } -1\le x,\ y<0 \text{ and } x^2+y^2>1 \end{cases}$	
• $\sin^{-1}x - \sin^{-1}y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & ,\ \text{if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \\ & \text{or} \\ & \text{if } xy>0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}, & \text{if } 0<x\le 1,\ -1\le y\le 0 \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}, & \text{if } -1\le x<0,\ 0<y\le 1 \text{ and } x^2+y^2>1 \end{cases}$	

8.	
• $\cos^{-1}x + \cos^{-1}y = \begin{cases} \cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} & ,\text{ if } -1\le x,\ y\le 1 \text{ and } x+y\ge 0 \\[2mm] 2\pi-\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\}, & \text{ if } -1\le x,\ y\le 1 \text{ and } x+y\le 0 \end{cases}$	
• $\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\} & ,\text{ if } -1\le x,\ y\le 1 \text{ and } x\le y \\[2mm] -\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\}, & \text{ if } -1\le y\le 0,\ 0<x\le 1 \text{ and } x\ge y \end{cases}$	

9.	
• $2\sin^{-1}x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}) & ,\text{ if } -\dfrac{1}{\sqrt{2}}\le x\le \dfrac{1}{\sqrt{2}} \\[2mm] \pi-\sin^{-1}(2x\sqrt{1-x^2}), & \text{ if } \dfrac{1}{\sqrt{2}}\le x\le 1 \\[2mm] -\pi-\sin^{-1}(2x\sqrt{1-x^2}) & ,\text{ if } -1\le x\le -\dfrac{1}{\sqrt{2}} \end{cases}$	

	$3\sin^{-1} x = \begin{cases} \sin^{-1}(3x - 4x^3) & , \text{ if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - \sin^{-1}(3x - 4x^3) & , \text{ if } \dfrac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x - 4x^3) & , \text{ if } -1 \le x < -\dfrac{1}{2} \end{cases}$	
10.	$2\cos^{-1} x = \begin{cases} \cos^{-1}(2x^2 - 1) & , \text{ if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2 - 1) & , \text{ if } -1 \le x \le 0 \end{cases}$	$3\cos^{-1} x = \begin{cases} \cos^{-1}(4x^3 - 3x) & , \text{ if } \dfrac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3 - 3x) & , \text{ if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 2\pi + \cos^{-1}(4x^3 - 3x) & , \text{ if } -1 \le x \le -\dfrac{1}{2} \end{cases}$
11.	$2\tan^{-1} x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1 - x^2}\right) & , \text{ if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1 - x^2}\right) & , \text{ if } x > 1 \\ -\pi + \tan^{-1}\left(\dfrac{2x}{1 - x^2}\right) & , \text{ if } x < -1 \end{cases}$	$3\tan^{-1} x = \begin{cases} \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) & , \text{ if } -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) & , \text{ if } x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) & , \text{ if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$
12.	$2\tan^{-1} x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right) & , \text{ if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right) & , \text{ if } x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right) & , \text{ if } x < -1 \end{cases}$	$2\tan^{-1} x = \begin{cases} \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right) & , \text{ if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right) & , \text{ if } -\infty < x \le 0 \end{cases}$
13.	$\sin^{-1} x = \cos^{-1}\sqrt{1 - x^2} = \tan^{-1}\dfrac{x}{\sqrt{1 - x^2}} = \cot^{-1}\dfrac{\sqrt{1 - x^2}}{x} = \sec^{-1}\left(\dfrac{1}{\sqrt{1 - x^2}}\right) = \text{cosec}^{-1}\left(\dfrac{1}{x}\right)$	
	$\cos^{-1} x = \sin^{-1}\sqrt{1 - x^2} = \tan^{-1}\left(\dfrac{\sqrt{1 - x^2}}{x}\right) = \cot^{-1}\left(\dfrac{x}{\sqrt{1 - x^2}}\right) = \sec^{-1}\dfrac{1}{x} = \text{cosec}^{-1}\left(\dfrac{1}{\sqrt{1 - x^2}}\right)$	
	$\tan^{-1} x = \sin^{-1}\left(\dfrac{x}{\sqrt{1 + x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1 + x^2}}\right) = \cot^{-1}\dfrac{1}{x} = \sec^{-1}\sqrt{1 + x^2} = \text{cosec}^{-1}\left(\dfrac{\sqrt{1 + x^2}}{x}\right)$	
14.	If $x_1, x_2, \ldots, x_n \in R$, then $\tan^{-1} x_1 + \tan^{-1} x_2 + \ldots + \tan^{-1} x_n = \tan^{-1}\left(\dfrac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}\right)$ where S_k = Sum of the products of $x_1, x_2, \ldots, x_n$ taken k at a time.	

Very Short Answer Type

1. Find the value of $\cos^{-1}\left(\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right)$.

2. Prove that : $\tan^{-1}\sqrt{x} = \dfrac{1}{2}\cos^{-1}\left(\dfrac{1-x}{1+x}\right),\ x \in [0,\ 1]$

3. Evaluate : $\sin\left\{\dfrac{\pi}{3} - \sin^{-1}\left(\dfrac{-1}{2}\right)\right\}$

4. Find the principal value of $\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right)$

5. Find the value of $\cos^2\left(\dfrac{1}{2}\cos^{-1}\left(\dfrac{3}{5}\right)\right)$.

Long Answer Type-I

6. Prove that : $\tan^{-1}\left(\dfrac{3}{4}\right) + \tan^{-1}\left(\dfrac{3}{5}\right) - \tan^{-1}\left(\dfrac{8}{19}\right) = \dfrac{\pi}{4}$.

7. Prove that : $4\,(\cot^{-1}3 + \operatorname{cosec}^{-1}\sqrt{5}) = \pi$.

8. Prove that : $\sin^{-1}\left(\dfrac{3}{5}\right) - \sin^{-1}\left(\dfrac{8}{17}\right) = \cos^{-1}\left(\dfrac{84}{85}\right)$.

9. Evaluate : $\tan\left[\dfrac{1}{2}\cos^{-1}\dfrac{\sqrt{5}}{3}\right]$.

10. Solve for x : $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\dfrac{3}{4}\right)$.

Long Answer Type-II

11. Prove that :
$$\tan\left(\dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right) + \tan\left(\dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right) = \dfrac{2b}{a}$$

12. Solve : $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$.

13. Prove that : $\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}] = \sqrt{\dfrac{x^2+1}{x^2+2}}$.

14. Solve : $\sin^{-1}x + \sin^{-1}2x = \dfrac{\pi}{3}$.

15. Prove that :
$$2\tan^{-1}\left(\dfrac{1}{5}\right) + \sec^{-1}\left(\dfrac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\dfrac{1}{8}\right) = \dfrac{\pi}{4}$$

SOLUTIONS

1. We have, $\cos^{-1}\left(\dfrac{1}{2}\right) = \cos^{-1}\left(\cos\dfrac{\pi}{3}\right) = \dfrac{\pi}{3}$

Also, $\sin^{-1}\left(\dfrac{1}{2}\right) = \sin^{-1}\left(\sin\dfrac{\pi}{6}\right) = \dfrac{\pi}{6}$

$\therefore\ \cos^{-1}\left(\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3} + \left(2 \times \dfrac{\pi}{6}\right) = \dfrac{2\pi}{3}$

2. $\tan^{-1}\sqrt{x} = \dfrac{1}{2}(2\tan^{-1}\sqrt{x}) = \dfrac{1}{2}\cos^{-1}\left(\dfrac{1-x}{1+x}\right)$

Hence proved.

3. We know that, $\sin^{-1}(-\theta) = -\sin^{-1}\theta$, for $\theta \in [-1,\ 1]$

$\therefore\ \sin\left\{\dfrac{\pi}{3} - \sin^{-1}\left(\dfrac{-1}{2}\right)\right\} = \sin\left\{\dfrac{\pi}{3} + \sin^{-1}\left(\dfrac{1}{2}\right)\right\}$

$= \sin\left(\dfrac{\pi}{3} + \sin^{-1}\left(\sin\dfrac{\pi}{6}\right)\right) = \sin\left(\dfrac{\pi}{3} + \dfrac{\pi}{6}\right) = \sin\dfrac{\pi}{2} = 1$.

4. $\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right) = \tan^{-1}\left\{\tan\left(\pi + \dfrac{\pi}{6}\right)\right\}$

$= \tan^{-1}\left\{\tan\left(\dfrac{\pi}{6}\right)\right\} = \dfrac{\pi}{6}\quad\left[\dfrac{\pi}{6} \in \left(-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right)\right]$

5. Let $\cos^{-1}\dfrac{3}{5} = \theta \Rightarrow \cos\theta = \dfrac{3}{5}$

$\therefore\ \cos^2\left\{\left(\dfrac{1}{2}\right)\cos^{-1}\left(\dfrac{3}{5}\right)\right\} = \cos^2\left(\dfrac{\theta}{2}\right)$

$= \dfrac{\cos\theta + 1}{2} = \dfrac{\dfrac{3}{5}+1}{2} = \dfrac{4}{5}$

6. We have, L.H.S.

$= \left\{\tan^{-1}\left(\dfrac{3}{4}\right) + \tan^{-1}\left(\dfrac{3}{5}\right)\right\} - \tan^{-1}\left(\dfrac{8}{19}\right)$

$= \tan^{-1}\left\{\dfrac{\dfrac{3}{4}+\dfrac{3}{5}}{1-\dfrac{3}{4}\times\dfrac{3}{5}}\right\} - \tan^{-1}\left(\dfrac{8}{19}\right)$

$= \tan^{-1}\left(\dfrac{27}{11}\right) - \tan^{-1}\left(\dfrac{8}{19}\right)$

$= \tan^{-1}\left\{\dfrac{\dfrac{27}{11}-\dfrac{8}{19}}{1+\dfrac{27}{11}\times\dfrac{8}{19}}\right\} = \tan^{-1}\left(\dfrac{425}{425}\right)$

$= \tan^{-1}1 = \dfrac{\pi}{4} = \text{R.H.S.}$

7. L.H.S. $= 4(\cot^{-1}3 + \operatorname{cosec}^{-1}\sqrt{5})$

$= 4\left(\tan^{-1}\left(\dfrac{1}{3}\right) + \sin^{-1}\left(\dfrac{1}{\sqrt{5}}\right)\right)$

$= 4\left[\tan^{-1}\left(\dfrac{1}{3}\right) + \tan^{-1}\left\{\dfrac{(1/\sqrt{5})}{\sqrt{1-(1/\sqrt{5})^2}}\right\}\right]$

$= 4\left[\tan^{-1}\left(\dfrac{1}{3}\right) + \tan^{-1}\left(\dfrac{1}{2}\right)\right]$

$$= 4\tan^{-1}\left[\dfrac{\dfrac{1}{3}+\dfrac{1}{2}}{1-\dfrac{1}{3}\times\dfrac{1}{2}}\right] = 4\tan^{-1}\left(\dfrac{5}{5}\right)$$

$$= 4\tan^{-1}(1) = 4\times\dfrac{\pi}{4} = \pi = \text{R.H.S.}$$

8. L.H.S. $= \sin^{-1}\left(\dfrac{3}{5}\right) - \sin^{-1}\left(\dfrac{8}{17}\right)$

$$= \sin^{-1}\left[\dfrac{3}{5}\sqrt{1-\left(\dfrac{8}{17}\right)^2} - \dfrac{8}{17}\sqrt{1-\left(\dfrac{3}{5}\right)^2}\right]$$

$$= \sin^{-1}\left[\dfrac{3}{5}\times\dfrac{15}{17} - \dfrac{8}{17}\times\dfrac{4}{5}\right]$$

$$= \sin^{-1}\left(\dfrac{45-32}{85}\right) = \sin^{-1}\left(\dfrac{13}{85}\right)$$

$$= \cos^{-1}\sqrt{1-\left(\dfrac{13}{85}\right)^2} = \cos^{-1}\left(\dfrac{84}{85}\right) = \text{R.H.S.}$$

9. Let $\cos^{-1}\left(\dfrac{\sqrt{5}}{3}\right) = 2\theta \Rightarrow \cos 2\theta = \dfrac{\sqrt{5}}{3}$ and $0\le 2\theta\le\pi$

Now, $\cos 2\theta = \dfrac{\sqrt{5}}{3}$

$$\therefore\quad \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{\sqrt{5}}{3} \Rightarrow \dfrac{2}{2\tan^2\theta} = \dfrac{3+\sqrt{5}}{3-\sqrt{5}}$$

<div style="text-align:center">[By componendo and dividendo]</div>

$$\Rightarrow\quad \tan^2\theta = \left(\dfrac{3-\sqrt{5}}{3+\sqrt{5}}\right)\left(\dfrac{3-\sqrt{5}}{3-\sqrt{5}}\right)$$

$$\Rightarrow\quad \tan^2\theta = \dfrac{(3-\sqrt{5})^2}{4} \Rightarrow \tan\theta = \pm\left(\dfrac{3-\sqrt{5}}{2}\right)$$

But $0\le 2\theta\le\pi$ $\therefore$ $0\le\theta\le\dfrac{\pi}{2}$

Hence $\tan\theta$ must be positive.

$$\therefore\quad \tan\theta = \dfrac{3-\sqrt{5}}{2}.$$

10. Let $\cot^{-1}\dfrac{3}{4} = \theta \Rightarrow \cot\theta = \dfrac{3}{4}$

$$\therefore\quad \csc\theta = \sqrt{1+\cot^2\theta} = \sqrt{1+\dfrac{9}{16}} = \dfrac{5}{4}$$

$$\therefore\quad \sin\theta = \dfrac{4}{5} \Rightarrow \theta = \sin^{-1}\dfrac{4}{5}$$

So, $\sin\left(\cot^{-1}\dfrac{3}{4}\right) = \sin\left(\sin^{-1}\dfrac{4}{5}\right) = \dfrac{4}{5}$

Let $\tan^{-1} x = \phi$. Then, $\tan\phi = x$

$$\therefore\quad \sec\phi = \sqrt{1+\tan^2\phi} = \sqrt{1+x^2}$$

$$\therefore\quad \cos\phi = \dfrac{1}{\sqrt{1+x^2}}$$

So, $\cos(\tan^{-1}x) = \cos\phi = \dfrac{1}{\sqrt{1+x^2}}$

Thus, $\dfrac{1}{\sqrt{1+x^2}} = \dfrac{4}{5} \Rightarrow \dfrac{1}{1+x^2} = \dfrac{16}{25} \Rightarrow 16x^2 = 9$

$$\Rightarrow\quad x^2 = \dfrac{9}{16} \Rightarrow x = \pm\dfrac{3}{4}$$

11. Let $\cos^{-1}\dfrac{a}{b} = \theta,\ \Rightarrow \cos\theta = \dfrac{a}{b}$

Now L.H.S. $= \tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right) + \tan\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)$

$$= \dfrac{1+\tan\dfrac{\theta}{2}}{1-\tan\dfrac{\theta}{2}} + \dfrac{1-\tan\dfrac{\theta}{2}}{1+\tan\dfrac{\theta}{2}} = \dfrac{\left(1+\tan\dfrac{\theta}{2}\right)^2 + \left(1-\tan\dfrac{\theta}{2}\right)^2}{1-\tan^2\dfrac{\theta}{2}}$$

$$= \dfrac{2\left(1+\tan^2\dfrac{\theta}{2}\right)}{1-\tan^2\dfrac{\theta}{2}} = \dfrac{2}{\cos\theta} = \dfrac{2}{a/b} = \dfrac{2b}{a} = \text{R.H.S.}$$

12. Given equation is $2\tan^{-1}(\cos x) = \tan^{-1}(2\csc x)$ ...(i)

Let $2\tan^{-1}(\cos x) = \theta \Rightarrow \cos x = \tan(\theta/2)$ ...(ii)

Now, $2\csc x = \dfrac{2}{\sin x} = \dfrac{2}{\sqrt{1-\cos^2 x}} = \dfrac{2}{\sqrt{1-\tan^2(\theta/2)}}$ ...(iii)

From (ii) and (iii), given equation becomes

$$\theta = \tan^{-1}\left[\dfrac{2}{\sqrt{1-\tan^2\left(\dfrac{\theta}{2}\right)}}\right] \Rightarrow \tan\theta = \dfrac{2}{\sqrt{1-\tan^2\left(\dfrac{\theta}{2}\right)}}$$

$$\Rightarrow\quad \dfrac{2\tan(\theta/2)}{1-\tan^2(\theta/2)} = \dfrac{2}{\sqrt{1-\tan^2(\theta/2)}}$$

$$\Rightarrow\quad \tan(\theta/2) = \sqrt{1-\tan^2(\theta/2)} \qquad\text{...(iv)}$$

$$\Rightarrow\quad \tan^2(\theta/2) = 1 - \tan^2(\theta/2) \Rightarrow 2\tan^2(\theta/2) = 1$$

$$\Rightarrow\quad \tan\dfrac{\theta}{2} = \dfrac{1}{\sqrt{2}} \qquad \left[\because \text{From (iv), } \tan\dfrac{\theta}{2}>0\right]$$

$$\Rightarrow\quad \cos x = \dfrac{1}{\sqrt{2}} \qquad\qquad\text{[From (ii)]}$$

$$\Rightarrow\quad \cos x = \cos\dfrac{\pi}{4} \qquad \therefore\quad x = 2n\pi\pm\dfrac{\pi}{4}, n\in Z$$

But for equation (i) to be satisfied cosec x and cos x must have same sign.

$\therefore$ x lies in 1st quadrant.

$\therefore$ $x = 2n\pi + \dfrac{\pi}{4}$, $n \in Z$

13. Let $\cot^{-1} x = \theta \Rightarrow x = \cot \theta$

$\therefore$ $\operatorname{cosec} \theta = \sqrt{1 + \cot^2 \theta} = \sqrt{1 + x^2}$

$\Rightarrow$ $\sin \theta = \dfrac{1}{\sqrt{1+x^2}} \Rightarrow \sin(\cot^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$

$\Rightarrow$ $\tan^{-1}\{\sin(\cot^{-1} x)\} = \tan^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) = \phi$ (say)

$\Rightarrow$ $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \cos \phi$ ...(i)

Now, $\tan^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) = \phi \Rightarrow \tan \phi = \dfrac{1}{\sqrt{1+x^2}}$

$\therefore$ $\sec \phi = \sqrt{1 + \tan^2 \phi} = \sqrt{1 + \dfrac{1}{(1+x^2)}} = \sqrt{\dfrac{2+x^2}{1+x^2}}$

$\Rightarrow$ $\cos \phi = \sqrt{\dfrac{1+x^2}{2+x^2}}$ ...(ii)

From (i) and (ii), we get

$\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \sqrt{\dfrac{x^2+1}{x^2+2}}$. Hence proved.

14. Given, $\sin^{-1} x + \sin^{-1} 2x = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

$\Rightarrow$ $\sin^{-1} x - \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = -\sin^{-1} 2x$ ...(i)

$\Rightarrow$ $\sin^{-1}\left[x\sqrt{1-\dfrac{3}{4}} - \dfrac{\sqrt{3}}{2}\sqrt{1-x^2}\right] = \sin^{-1}(-2x)$

$\Rightarrow$ $\dfrac{x}{2} - \dfrac{\sqrt{3}}{2}\sqrt{1-x^2} = -2x$

$\Rightarrow$ $5x = \sqrt{3}\sqrt{1-x^2}$ ...(ii)

On squaring, we get $25x^2 = 3(1 - x^2) \Rightarrow 28x^2 = 3$

$\therefore$ $x = \pm\dfrac{\sqrt{3}}{2\sqrt{7}}$

But $x = -\dfrac{\sqrt{3}}{2\sqrt{7}}$ does not satisfy the equation as negative value of x makes L.H.S. of the equation (ii) negative where as R.H.S. is positive.

$\therefore$ $x = \dfrac{\sqrt{3}}{2\sqrt{7}} = \dfrac{1}{2}\sqrt{\dfrac{3}{7}}$.

15. $\sec^{-1}\left(\dfrac{5\sqrt{2}}{7}\right) = \cos^{-1}\left(\dfrac{7}{5\sqrt{2}}\right)$

$= \tan^{-1}\left(\dfrac{\sqrt{1 - \left(\dfrac{7}{5\sqrt{2}}\right)^2}}{\dfrac{7}{5\sqrt{2}}}\right) = \tan^{-1}\left(\dfrac{1}{7}\right)$

$\therefore$ L.H.S. $= 2\left(\tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{8}\right) + \tan^{-1}\dfrac{1}{7}$

$= 2\cdot\tan^{-1}\left\{\dfrac{\dfrac{1}{5}+\dfrac{1}{8}}{1 - \dfrac{1}{5}\times\dfrac{1}{8}}\right\} + \tan^{-1}\dfrac{1}{7}$

$= 2\tan^{-1}\left(\dfrac{13}{39}\right) + \tan^{-1}\dfrac{1}{7}$

$= 2\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{7}$

$= \tan^{-1}\left\{\dfrac{2\times\dfrac{1}{3}}{1 - \left(\dfrac{1}{3}\right)^2}\right\} + \tan^{-1}\dfrac{1}{7}$

$= \tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{1}{7}$

$= \tan^{-1}\left\{\dfrac{\dfrac{3}{4}+\dfrac{1}{7}}{1 - \dfrac{3}{4}\times\dfrac{1}{7}}\right\} = \tan^{-1}\left(\dfrac{25}{25}\right)$

$= \tan^{-1}(1) = \dfrac{\pi}{4} =$ R.H.S.

Kerala PET

SOLVED PAPER 2016

1. If $[1 \ x \ 1] \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$, then the values of x are
(a) 1, 5   (b) –1, –5   (c) 1, 6
(d) –1, –6   (e) 3, 3

2. If $A = \begin{vmatrix} 8 & 27 & 125 \\ 2 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix}$, then the value of A^2 is equal to
(a) 0   (b) 36   (c) 64
(d) 2400   (e) 3600

3. If $A = \begin{bmatrix} x & 1 & -x \\ 0 & 1 & -1 \\ x & 0 & 7 \end{bmatrix}$ and $\det(A) = \begin{vmatrix} 3 & 0 & 1 \\ 2 & -1 & 2 \\ 0 & 0 & 3 \end{vmatrix}$, then the value of x is
(a) –3   (b) 3   (c) 2
(d) –8   (e) –2

4. The coefficient of x^2 in the expansion of the determinant $\begin{vmatrix} x^2 & x^3+1 & x^5+2 \\ x^3+3 & x^2+x & x^3+x^4 \\ x+4 & x^3+x^5 & 2^3 \end{vmatrix}$ is
(a) –10   (b) –8   (c) –2
(d) –6   (e) 8

5. Let $A = \begin{bmatrix} 1 & \dfrac{-1-i\sqrt{3}}{2} \\ \dfrac{-1+i\sqrt{3}}{2} & 1 \end{bmatrix}$. Then $A^{100} =$
(a) $2^{100} A$ (b) $2^{99} A$ (c) $2^{98} A$
(d) A (e) A^2

6. The least integer satisfying $\dfrac{396}{10} - \dfrac{19-x}{10} < \dfrac{376}{10} - \dfrac{19-9x}{10}$ is
(a) 1   (b) 2   (c) 3
(d) 4   (e) 5

7. If $|x - 1| + |x - 3| \le 8$, then the values of x lie in the interval
(a) $(-\infty, -2]$ (b) $[-2, 6]$ (c) $(-3, 7)$
(d) $(-2, \infty)$ (e) $[6, \infty)$

8. Let p : 57 is an odd prime number,
q : 4 is a divisor of 12,
r = 15 is the LCM of 3 and 5
be three simple logical statements. Which one of the following is true?
(a) $p \vee (\sim q \wedge r)$ (b) $\sim p \vee (q \wedge r)$
(c) $(p \wedge q) \vee \sim r$ (d) $(p \vee q) \wedge \sim r$
(e) $\sim p \wedge (\sim q \wedge r)$

9. Let p, q, r be three simple statements.
Then $\sim (p \vee q) \vee \sim (p \vee r) \equiv$
(a) $(\sim p) \wedge (\sim q \vee \sim r)$
(b) $(\sim p) \wedge (q \vee r)$ (c) $p \wedge (q \vee r)$
(d) $p \vee (q \wedge r)$ (e) $(p \vee q) \wedge r$

10. If p : 3 is a prime number and q : one plus one is three, then the compound statement "It is not that 3 is a prime number or it is not that one plus one is three" is
(a) $\sim p \vee q$ (b) $\sim (p \vee q)$ (c) $p \wedge \sim q$
(d) $\sim p \vee \sim q$ (e) $p \vee \sim q$

11. The value of $\sin^2 \dfrac{\pi}{8} + \sin^2 \dfrac{3\pi}{8} + \sin^2 \dfrac{5\pi}{8} + \sin^2 \dfrac{7\pi}{8}$ is equal to
(a) $\dfrac{1}{8}$ (b) $\dfrac{1}{4}$ (c) $\dfrac{1}{2}$
(d) 1   (e) 2

12. The value of $\dfrac{\sqrt{3}}{\sin 15°} - \dfrac{1}{\cos 15°}$ is equal to
(a) $4\sqrt{2}$ (b) $2\sqrt{2}$ (c) $\sqrt{2}$
(d) $\dfrac{1}{\sqrt{2}}$ (e) $\dfrac{\sqrt{3}}{2}$

13. If $\sin x + \cos x = \sqrt{2}$, then $\sin x \cos x =$
(a) 1   (b) $\dfrac{1}{2}$ (c) 2

(d) $\sqrt{2}$ (e) $\dfrac{1}{\sqrt{2}}$

14. If $\tan\theta = \dfrac{1}{2}$ and $\tan\phi = \dfrac{1}{3}$, then $\tan(2\theta + \phi) =$

(a) $\dfrac{3}{4}$ (b) $\dfrac{4}{3}$ (c) $\dfrac{1}{3}$

(d) 3 (e) $\dfrac{1}{2}$

15. The value of x satisfying the equation

$\tan^{-1}x + \tan^{-1}\left(\dfrac{2}{3}\right) = \tan^{-1}\left(\dfrac{7}{4}\right)$ is equal to

(a) $\dfrac{1}{2}$ (b) $-\dfrac{1}{2}$ (c) $\dfrac{3}{2}$

(d) $-\dfrac{1}{3}$ (e) $\dfrac{1}{3}$

16. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then $\cot(A - B)$ is

(a) $\dfrac{1}{x - y}$ (b) $\dfrac{1}{x + y}$ (c) $\dfrac{1}{x} + y$

(d) $\dfrac{1}{x} - \dfrac{1}{y}$ (e) $\dfrac{1}{x} + \dfrac{1}{y}$

17. If $\tan^{-1}x + \tan^{-1}y = \dfrac{2\pi}{3}$, then $\cot^{-1}x + \cot^{-1}y$ is equal to

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{\pi}{3}$

(d) $\dfrac{\sqrt{3}}{2}$ (e) π

18. If the orthocenter, centroid, incentre and circumcentre coincide in a triangle ABC, and if the length of side AB is $\sqrt{75}$, units then the length of the altitude of the triangle through the vertex A is

(a) $\sqrt{3}$ units   (b) 3 units   (c) $\dfrac{\sqrt{15}}{2}$ units

(d) $\dfrac{15}{2}$ units   (e) $\sqrt{\dfrac{5}{2}}$ units

19. If $A(2, 4)$ and $B(6, 10)$ are two fixed points and if a point P moves so that $\angle APB$ is always a right angle, then the locus of P is

(a) $x^2 + y^2 + 8x + 14y + 52 = 0$
(b) $x^2 + y^2 - 8x + 14y - 52 = 0$
(c) $x^2 + y^2 + 8x - 14y + 52 = 0$
(d) $x^2 + y^2 - 8x - 14y - 52 = 0$
(e) $x^2 + y^2 - 8x - 14y + 52 = 0$

20. The points $(-1, 0)$ and $(-2, 1)$ are the two extremities of a diagonal of a parallelogram. If $(-6, 5)$ is the third vertex, then the fourth vertex of the parallelogram is
(a) $(2, -6)$ (b) $(2, -5)$ (c) $(3, -4)$
(d) $(-3, 4)$ (e) $(3, -5)$

21. The slope of the straight line $\dfrac{x}{10} - \dfrac{y}{4} = 3$ is

(a) $\dfrac{5}{2}$ (b) $\dfrac{-5}{2}$ (c) $\dfrac{2}{5}$

(d) $\dfrac{-2}{5}$ (e) $\dfrac{3}{4}$

22. If y-intercept of the line $4x - ay = 8$ is thrice its x-intercept, then the value of a is equal to

(a) $\dfrac{3}{4}$ (b) $\dfrac{4}{3}$ (c) $-\dfrac{3}{4}$

(d) $-\dfrac{4}{3}$ (e) $-\dfrac{2}{3}$

23. The equation of one of the straight lines passing through the point $(0, 1)$ and is at a distance of $\dfrac{3}{5}$ units from the origin is
(a) $4x + 3y = 3$ (b) $-x + y = 1$ (c) $x + y = 1$
(d) $5x + 4y = 4$ (e) $-5x + 4y = 4$

24. The nearest point on the line $x + y - 3 = 0$ from the point $(3, -2)$ is
(a) $(3, 5)$ (b) $(4, 1)$ (c) $(3, -5)$
(d) $(4, -1)$ (e) $(5, -1)$

25. The image of the origin with respect to the line $4x + 3y = 25$, is
(a) $(4, 3)$ (b) $(3, 4)$ (c) $(6, 8)$
(d) $(4, 6)$ (e) $(8, 6)$

26. If the area of the circle $4x^2 + 4y^2 + 8x - 16y + \lambda = 0$ is 9π sq. units, then the value of λ is
(a) 4 (b) -4 (c) 16
(d) -16 (e) -8

27. The radius of the circle passing through the points $(2, 3)$, $(2, 7)$ and $(5, 3)$ is

(a) 5 (b) 4 (c) $\dfrac{5}{2}$
(d) 2 (e) $\sqrt{5}$

28. If a diameter of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord of another circle C having centre $(2, 1)$, then the radius of the circle C is
(a) 2 (b) $\sqrt{3}$ (c) 3
(d) $\sqrt{5}$ (e) 5

29. In the family of concentric circles $2(x^2 + y^2) = k$, the radius of the circle passing through $(1, 1)$ is
(a) $\sqrt{2}$ (b) 4 (c) $2\sqrt{2}$
(d) 1 (e) $3\sqrt{2}$

30. Let P be a point on an ellipse at a distance of 8 units from a focus. If the eccentricity is $\dfrac{4}{5}$, then the distance of the point P from the directrix is
(a) $\dfrac{5}{8}$ (b) $\dfrac{8}{5}$ (c) 5
(d) 8 (e) 10

31. If $(-3, 0)$ is the vertex and y-axis is the directrix of a parabola, then its focus is at the point
(a) $(0, -6)$ (b) $(-6, 0)$ (c) $(6, 0)$
(d) $(0, 0)$ (e) $(3, 0)$

32. The foci of the ellipse $4x^2 + 9y^2 = 1$ are
(a) $\left(\pm\dfrac{\sqrt{3}}{2}, 0\right)$ (b) $\left(\pm\dfrac{\sqrt{5}}{2}, 0\right)$ (c) $\left(\pm\dfrac{\sqrt{5}}{3}, 0\right)$
(d) $\left(\pm\dfrac{\sqrt{5}}{6}, 0\right)$ (e) $\left(\pm\dfrac{\sqrt{5}}{4}, 0\right)$

33. The directrix of a parabola is $x + 8 = 0$ and its focus is at $(4, 3)$. Then the length of the latus-rectum of the parabola is
(a) 5 (b) 9 (c) 10
(d) 12 (e) 24

34. If the eccentricity of the ellipse $ax^2 + 4y^2 = 4a$, $(a < 4)$ is $\dfrac{1}{\sqrt{2}}$, then its semi-minor axis is equal to
(a) 2 (b) $\sqrt{2}$ (c) 1
(d) $\sqrt{3}$ (e) 3

35. The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ passes through the point $(\sqrt{6}, 3)$ and the length of the latus rectum is $\dfrac{18}{5}$. Then the length of the transverse axis is equal to
(a) 5 (b) 4 (c) 3
(d) 2 (e) 1

36. The angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{5\pi}{6}$ and the projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{-9}{\sqrt{3}}$, then $|\vec{a}|$ is equal to
(a) 12 (b) 8 (c) 10
(d) 4 (e) 6

37. The direction cosines of the straight line given by the planes $x = 0$ and $z = 0$ are
(a) $1, 0, 0$ (b) $0, 0, 1$ (c) $1, 1, 0$
(d) $0, 1, 0$ (e) $0, 1, 1$

38. If $\vec{a} = 2\hat{i} - \hat{j} - m\hat{k}$ and $\vec{b} = \dfrac{4}{7}\hat{i} - \dfrac{2}{7}\hat{j} + 2\hat{k}$ are collinear, then the value of m is equal to
(a) -7 (b) -1 (c) 2
(d) 7 (e) -2

39. Let $\vec{a} = 2\hat{i} + 5\hat{j} - 7\hat{k}$, $\vec{b} = \hat{i} + 3\hat{j} + 5\hat{k}$. Then $(3\vec{a} - 5\vec{b})\cdot(4\vec{a} \times 5\vec{b}) = $
(a) -7 (b) 0 (c) -13
(d) 1 (e) -8

40. If $\vec{a} + 2\vec{b} - \vec{c} = \vec{0}$ and $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \lambda \vec{a} \times \vec{b}$, then the value of λ is equal to
(a) 5 (b) 4 (c) 2
(d) -2 (e) -4

41. If $\vec{a}\cdot\vec{b} = 0$ and $\vec{a} + \vec{b}$ makes an angle of 60° with $\vec{b}$, then $|\vec{a}|$ is equal to
(a) 0 (b) $\dfrac{1}{\sqrt{3}}|\vec{b}|$ (c) $\dfrac{1}{|\vec{b}|}$
(d) $|\vec{b}|$ (e) $\sqrt{3}|\vec{b}|$

42. If $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular and $\vec{b} = 3\hat{i} - 4\hat{j} + 2\hat{k}$, then $|\vec{a}|$ is equal to
(a) $\sqrt{41}$ (b) $\sqrt{39}$ (c) $\sqrt{19}$
(d) $\sqrt{29}$ (e) $\sqrt{31}$

43. The straight line $\vec{r} = (\hat{i} + \hat{j} + \hat{k}) + \alpha(2\hat{i} - \hat{j} + 4\hat{k})$ meets the xy plane at the point
(a) $(2, -1, 0)$ (b) $(3, 4, 0)$ (c) $\left(\dfrac{1}{2}, \dfrac{3}{4}, 0\right)$
(d) $\left(\dfrac{1}{2}, \dfrac{7}{4}, 0\right)$ (e) $\left(\dfrac{1}{2}, \dfrac{5}{4}, 0\right)$

44. The equation of the plane passing through $(-1, 5, -7)$ and parallel to the plane $2x - 5y + 7z + 11 = 0$, is
(a) $\vec{r}\cdot(2\hat{i} - 5\hat{j} - 7\hat{k}) + 76 = 0$
(b) $\vec{r}\cdot(2\hat{i} - 5\hat{j} + 7\hat{k}) + 76 = 0$
(c) $\vec{r}\cdot(2\hat{i} - 5\hat{j} + 7\hat{k}) + 75 = 0$
(d) $\vec{r}\cdot(2\hat{i} - 5\hat{j} + 7\hat{k}) + 65 = 0$
(e) $\vec{r}\cdot(2\hat{i} - 5\hat{j} - 7\hat{k}) + 55 = 0$

45. The angle subtended at the point $(1, 2, 3)$ by the points $P(2, 4, 5)$ and $Q(3, 3, 1)$, is

(a) 90°   (b) 60°   (c) 30°
(d) 0°   (e) 45°

46. If the two lines $\dfrac{x-1}{2} = \dfrac{1-y}{-a} = \dfrac{z}{4}$ and

$\dfrac{x-3}{1} = \dfrac{2y-3}{4} = \dfrac{z-2}{2}$ are perpendicular, then the value of a is equal to
(a) −4   (b) 5   (c) −5
(d) 4   (e) −2

47. If the line $\dfrac{x+1}{2} = \dfrac{y+1}{3} = \dfrac{z+1}{4}$ meets the plane $x + 2y + 3z = 14$ at P, then the distance between P and the origin is
(a) $\sqrt{14}$ (b) $\sqrt{15}$ (c) $\sqrt{13}$
(d) $\sqrt{12}$ (e) $\sqrt{17}$

48. The point of intersection of the straight lines
$\vec{r} = (3\hat{i} - 4\hat{j} + 5\hat{k}) + \lambda(-\hat{i} - 2\hat{j} + 2\hat{k})$ and
$\dfrac{3-x}{-1} = \dfrac{y+4}{2} = \dfrac{z-5}{7}$ is
(a) (−3, −4, −5) (b) (−3, 4, 5)   (c) (−3, 4, −5)
(d) (−3, −4, 5) (e) (3, −4, 5)

49. The vector equation of the straight line
$\dfrac{x-2}{1} = \dfrac{y}{-3} = \dfrac{1-z}{2}$ is
(a) $\vec{r} = 2\hat{i} + \hat{k} + t(\hat{i} + 3\hat{j} + 2\hat{k})$
(b) $\vec{r} = 2\hat{i} - \hat{k} + t(\hat{i} - 3\hat{j} - 2\hat{k})$
(c) $\vec{r} = 2\hat{i} + \hat{k} + t(\hat{i} - 3\hat{j} + 2\hat{k})$
(d) $\vec{r} = 2\hat{i} - \hat{j} + t(\hat{i} - 3\hat{j} - 2\hat{k})$
(e) $\vec{r} = 2\hat{i} + \hat{k} + t(\hat{i} - 3\hat{j} - 2\hat{k})$

50. The straight line $\vec{r} = (\hat{i} + \hat{j} + 2\hat{k}) + t(2\hat{i} + 5\hat{j} + 3\hat{k})$ is parallel to the plane $\vec{r} \cdot (2\hat{i} + \hat{j} - 3\hat{k}) = 5$. Then the distance between the straight line and the plane is
(a) $\dfrac{9}{\sqrt{14}}$ (b) $\dfrac{8}{\sqrt{14}}$ (c) $\dfrac{7}{\sqrt{14}}$
(d) $\dfrac{6}{\sqrt{14}}$ (e) $\dfrac{5}{\sqrt{14}}$

51. Two fair dice are rolled. Then the probability of getting a composite number as the sum of face values is equal to
(a) $\dfrac{7}{12}$ (b) $\dfrac{5}{12}$ (c) $\dfrac{1}{12}$
(d) $\dfrac{3}{4}$ (e) $\dfrac{2}{3}$

52. If the mean of the numbers a, b, 8, 5, 10 is 6 and their variance is 6.8, then ab is equal to
(a) 6   (b) 7   (c) 12
(d) 14   (e) 25

53. In a class, in an examination in Mathematics, 10 students scored 100 marks each, 2 students scored zero and the average of the remaining students is 72 marks. If the class average is 76, then the number of students in the class is
(a) 44   (b) 40   (c) 38
(d) 34   (e) 32

54. A bag contains 3 red, 4 white and 5 blue balls. If two balls are drawn at random, then the probability that they are of different colours is
(a) $\dfrac{47}{66}$ (b) $\dfrac{23}{33}$ (c) $\dfrac{47}{132}$
(d) $\dfrac{47}{33}$ (e) $\dfrac{70}{33}$

55. There are 5 positive numbers and 6 negative numbers. Three numbers are chosen at random and multiplied. The probability that the product being a negative number is
(a) $\dfrac{11}{34}$ (b) $\dfrac{17}{33}$ (c) $\dfrac{16}{35}$
(d) $\dfrac{15}{34}$ (e) $\dfrac{16}{33}$

56. The value of $\displaystyle\lim_{x \to 0} \dfrac{\cot 4x}{\csc 3x}$ is equal to
(a) $\dfrac{4}{3}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{2}{3}$
(d) $\dfrac{3}{2}$ (e) 0

57. Let $f(x) = \begin{cases} \cos x & \text{if } x \geq 0 \\ -\cos x & \text{if } x < 0 \end{cases}$.

Which one of the following statements is **not true**?
(a) $f(x)$ is continuous at $x = 1$
(b) $f(x)$ is continuous at $x = -1$
(c) $f(x)$ is continuous at $x = 2$
(d) $f(x)$ is continuous at $x = -2$
(e) $f(x)$ is continuous at $x = 0$

58. The value of $\displaystyle\lim_{n \to \infty} \dfrac{{}^nC_3 - {}^nP_3}{n^3}$ is equal to
(a) $\dfrac{-5}{6}$ (b) $\dfrac{5}{6}$ (c) $\dfrac{1}{6}$
(d) $-\dfrac{1}{6}$ (e) $\dfrac{2}{3}$

59. If $f(x) = 3x + 5$ and $g(x) = x^2 - 1$, then $(fog)(x^2 - 1)$ is equal to
(a) $3x^4 - 3x + 5$ (b) $3x^4 - 6x^2 + 5$
(c) $6x^4 + 3x^2 + 5$ (d) $6x^4 - 6x + 5$
(e) $3x^2 + 6x + 4$

60. The period of the function $f(x) = \tan(4x - 1)$ is
(a) π (b) $\dfrac{\pi}{2}$ (c) 2π
(d) $\dfrac{\pi}{4}$ (e) $\dfrac{3\pi}{4}$

61. If $2^x + 2^y = 2^{x+y}$, then the value of $\dfrac{dy}{dx}$ at $(1, 1)$ is equal to
(a) -2 (b) -1 (c) 0
(d) 1 (e) 2

62. If $f(x) = \dfrac{\sin^{-1} x}{\sqrt{1 - x^2}}$, then the value of $(1 - x^2) f'(x) - xf(x)$ is
(a) 0 (b) 1 (c) 2
(d) 3 (e) 4

63. If $f(x) = \left(\dfrac{x}{2}\right)^{10}$, then $f(1) + \dfrac{f'(1)}{\lfloor 1} + \dfrac{f''(1)}{\lfloor 2} + \dfrac{f'''(1)}{\lfloor 3} + \dots + \dfrac{f^{(10)}(1)}{\lfloor 10}$ is equal to
(a) 1 (b) 10 (c) 11
(d) 512 (e) 1024

64. If $f'(4) = 5$, $g'(4) = 12$, $f(4)\, g(4) = 2$ and $g(4) = 6$, then $\left(\dfrac{f}{g}\right)'(4) =$
(a) $\dfrac{5}{36}$ (b) $\dfrac{11}{18}$ (c) $\dfrac{23}{36}$
(d) $\dfrac{13}{18}$ (e) $\dfrac{19}{36}$

65. If the derivative of $(ax - 5)e^{3x}$ at $x = 0$ is -13, then the value of a is equal to
(a) 8 (b) -5 (c) 5
(d) -2 (e) 2

66. Let $y = \tan^{-1}(\sec x + \tan x)$. Then $\dfrac{dy}{dx} =$
(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{\sec x + \tan x}$
(d) $\dfrac{1}{\sec^2 x}$ (e) $\dfrac{1}{\tan x}$

67. If $s = \sec^{-1}\left(\dfrac{1}{2x^2 - 1}\right)$ and $t = \sqrt{1 - x^2}$, then $\dfrac{ds}{dt}$ at $x = \dfrac{1}{2}$ is
(a) 1 (b) 2 (c) -2
(d) 4 (e) -4

68. The minimum value of $2x^3 - 9x^2 + 12x + 4$ is
(a) 4 (b) 5 (c) 6
(d) 7 (e) 8

69. The slope of the curve $y = e^x \cos x$, $x \in (-\pi, \pi)$ is maximum at
(a) $x = \dfrac{\pi}{2}$ (b) $x = -\dfrac{\pi}{2}$ (c) $x = \dfrac{\pi}{4}$
(d) $x = 0$ (e) $x = \dfrac{\pi}{3}$

70. If $y = f(x)$ is continuous on $[0, 6]$, differentiable on $(0, 6)$, $f(0) = -2$ and $f(6) = 16$, then at some point between $x = 0$ and $x = 6$, $f'(x)$ must be equal to
(a) -18 (b) -3 (c) 3
(d) 14 (e) 18

71. The equation of the tangent to the curve $y = x^3 - 6x + 5$ at $(2, 1)$ is
(a) $6x - y - 11 = 0$ (b) $6x - y - 13 = 0$
(c) $6x + y + 11 = 0$ (d) $6x - y + 11 = 0$
(e) $x - 6y - 11 = 0$

72. Let $f(x) = 2x^3 - 5x^2 - 4x + 3$, $\dfrac{1}{2} \le x \le 3$. The point at which the tangent to the curve is parallel to the x-axis, is
(a) $(1, -4)$ (b) $(2, -9)$ (c) $(2, -4)$
(d) $(2, -1)$ (e) $(2, -5)$

73. Two sides of a triangle are 8 m and 5 m in length. The angle between them is increasing at the rate 0.08 rad/sec. When the angle between the sides of fixed length is $\dfrac{\pi}{3}$, the rate at which the area of the triangle is increasing is,
(a) 0.4 m^2/sec   (b) 0.8 m^2/sec
(c) 0.6 m^2/sec   (d) 0.04 m^2/sec
(e) 0.08 m^2/sec

74. If $y = 8x^3 - 60x^2 + 144x + 27$ is a strictly decreasing function in the interval
(a) $(-5, 6)$ (b) $(-\infty, 2)$ (c) $(5, 6)$
(d) $(3, \infty)$ (e) $(2, 3)$

75. $\int (\sec x)^m (\tan^3 x + \tan x)\,dx$ is equal to

(a) $\sec^{m+2} x + C$ (b) $\tan^{m+2} x + C$

(c) $\dfrac{\sec^{m+2} x}{m+2} + C$ (d) $\dfrac{\tan^{m+2} x}{m+2} + C$

(e) $\dfrac{\sec^{m+1} x}{m+1} + C$

76. $\int \dfrac{1}{7}\sin\left(\dfrac{x}{7}+10\right)dx$ is equal to

(a) $\dfrac{1}{7}\cos\left(\dfrac{x}{7}+10\right)+C$ (b) $-\dfrac{1}{7}\cos\left(\dfrac{x}{7}+10\right)+C$

(c) $-\cos\left(\dfrac{x}{7}+10\right)+C$ (d) $-7\cos\left(\dfrac{x}{7}+10\right)+C$

(e) $\cos(x+70)+C$

77. $\int\left(\dfrac{x-a}{x}-\dfrac{x}{x+a}\right)dx$ is equal to

(a) $\log\left|\dfrac{x+a}{x}\right|+C$ (b) $a\log\left|\dfrac{x+a}{x}\right|+C$

(c) $a\log\left|\dfrac{x}{x+a}\right|+C$ (d) $\log\left|\dfrac{x}{x+a}\right|+C$

(e) $a\log\left|\dfrac{x-a}{x+a}\right|+C$

78. $\int x^4 e^{x^5}\cos\left(e^{x^5}\right)dx$ is equal to

(a) $\dfrac{1}{3}\sin\left(e^{x^5}\right)+C$ (b) $\dfrac{1}{4}\sin\left(e^{x^5}\right)+C$

(c) $\dfrac{1}{5}\sin\left(e^{x^5}\right)+C$ (d) $\sin\left(e^{x^5}\right)+C$

(e) $2\sin\left(e^{x^5}\right)+C$

79. $\int \dfrac{2x+\sin 2x}{1+\cos 2x}dx$ is equal to

(a) $x+\log|\tan x|+C$ (b) $x\log|\tan x|+C$

(c) $x\tan x+C$ (d) $x+\tan x+C$

(e) $x\sec x+C$

80. $\int \dfrac{1}{\sin x\cos x}dx$ is equal to

(a) $\log|\tan x|+C$ (b) $\log|\sin 2x|+C$

(c) $\log|\sec x|+C$ (d) $\log|\cos x|+C$

(e) $\log|\sin x|+C$

81. $\int \dfrac{1}{8\sin^2 x+1}dx$ is equal to

(a) $\sin^{-1}(\tan x)+C$ (b) $\dfrac{1}{3}\sin^{-1}(\tan x)+C$

(c) $\dfrac{1}{3}\tan^{-1}(3\tan x)+C$

(d) $\tan^{-1}(3\tan x)+C$ (e) $\sin^{-1}(3\tan x)+C$

82. $\displaystyle\int_0^{\pi/2} \log\left(\dfrac{\cos x}{\sin x}\right)dx$ is equal to

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{4}$ (c) π

(d) 2π (e) 0

83. The value of $\displaystyle\int_{-1}^{2} 4x^2 \,|\,x\,|\, dx$ is equal to

(a) 17   (b) 16   (c) 15

(d) 14   (e) 13

84. The area of the region bounded by $y^2 = 16 - x^2$, $y = 0$, $x = 0$ in the first quadrant is (in square units)

(a) 8π (b) 6π (c) 2π

(d) 4π (e) $\dfrac{\pi}{2}$

85. The value of $\displaystyle\int_2^4 (x-2)(x-3)(x-4)\,dx$ is equal to

(a) $\dfrac{1}{2}$ (b) 2   (c) 3

(d) $\dfrac{1}{3}$ (e) 0

86. The area bounded by the lines $y - 2x = 2$, $y = 4$ and the y-axis is equal to (in square units)

(a) 1   (b) 4   (c) 0

(d) 3   (e) 2

87. The general solution of the differential equation

$(x+y+3)\dfrac{dy}{dx}=1$ is

(a) $x+y+3 = Ce^y$ (b) $x+y+4 = Ce^y$

(c) $x+y+3 = Ce^{-y}$ (d) $x+y+4 = Ce^{-y}$

(e) $x+y+4e^y = C$

88. The differential equation representing the family of curves $y^2 = a(ax+b)$ where a and b are arbitrary constants, is of

(a) order 1, degree 1   (b) order 1, degree 3

(c) order 2, degree 3   (d) order 1, degree 4

(e) order 2, degree 1

89. The solution of the differential equation

$\dfrac{x\dfrac{dy}{dx}-y}{\sqrt{x^2-y^2}}=10x^2$ is

(a) $\sin^{-1}\left(\dfrac{y}{x}\right) - 5x^2 = C$

(b) $\sin^{-1}\left(\dfrac{y}{x}\right) = 10x^2 + C$

(c) $\dfrac{y}{x} = 5x^2 + C$

(d) $\sin^{-1}\left(\dfrac{y}{x}\right) = 10x^2 + Cx$

(e) $\sin^{-1}\left(\dfrac{y}{x}\right) + 5x^2 = C$

90. The general solution of the differential equation $x\,dy - y\,dx = y^2dx$ is

(a) $y = \dfrac{x}{C - x}$ (b) $x = \dfrac{2y}{C + x}$

(c) $y = (C + x)\,(2x)$ (d) $y = \dfrac{2x}{C + x}$

(e) $x = \dfrac{y}{C - x}$

91. If $*$ is the operation defined by $a * b = a^b$ for $a, b \in N$, then $(2 * 3) * 2$ is equal to
(a) 81   (b) 512   (c) 216
(d) 64   (e) 243

92. The domain of the function
$$f(x) = \begin{cases} \dfrac{(x^2 - 9)}{(x - 3)}, & \text{if } x \neq 3 \\ 6, & \text{if } x = 3 \end{cases} \text{ is}$$
(a) $(0, 3)$ (b) $(-\infty, 3)$ (c) $(-\infty, \infty)$
(d) $(3, \infty)$ (e) $(-3, 3)$

93. Let $f(x) = x^3$ and $g(x) = 3^x$. The values of a such that $g(f(a)) = f(g(a))$ are
(a) 0, 2   (b) 1, 3   (c) $0, \pm 3$
(d) $1, \pm 2$ (e) $0, \pm\sqrt{3}$

94. If $f\left(\dfrac{x+1}{2x-1}\right) = 2x$, $x \in N$, then the value of $f(2)$ is equal to
(a) 1   (b) 4   (c) 3
(d) 2   (e) 5

95. If $A \setminus B = \{a, b\}$, $B \setminus A = \{c, d\}$ and $A \cap B = \{e, f\}$, then the set B is equal to
(a) $\{a, b, c, d\}$ (b) $\{e, f, c, d\}$ (c) $\{a, b, e, f\}$
(d) $\{c, d, a, e\}$ (e) $\{d, e, a, b\}$

96. The function $f : A \to B$ given by $f(x) = x$, $x \in A$, is one to one but not onto. Then

(a) $B \subset A$ (b) $A = B$ (c) $A' \subset B'$
(d) $A \subset B$ (e) $A \cap B = \varnothing$

97. The principal argument of the complex number
$$z = \dfrac{1 + \sin\dfrac{\pi}{3} + i\cos\dfrac{\pi}{3}}{1 + \sin\dfrac{\pi}{3} - i\cos\dfrac{\pi}{3}} \text{ is}$$
(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{6}$ (c) $\dfrac{2\pi}{3}$

(d) $\dfrac{\pi}{2}$ (e) $\dfrac{\pi}{4}$

98. If $\dfrac{(1+i)(2+3i)(3-4i)}{(2-3i)(1-i)(3+4i)} = a + ib$, then $a^2 + b^2 =$
(a) 132   (b) 25   (c) 144
(d) 128   (e) 1

99. Let z, w be two non-zero complex numbers. If $\overline{z} + iw = 0$ and $\arg(zw) = \pi$, then $\arg z =$
(a) π (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{4}$

(d) $\dfrac{\pi}{6}$ (e) $\dfrac{\pi}{8}$

100. If $z = \dfrac{2-i}{i}$, then $\text{Re}(z^2) + \text{Im}(z^2)$ is equal to
(a) 1   (b) –1   (c) 2
(d) –2   (e) 3

101. If $|z + 1| < |z - 1|$, then z lies
(a) on the x-axis   (b) on the y-axis
(c) in the region $x < 0$ (d) in the region $y > 0$
(e) in the region $x > y$

102. If $\left|z - \dfrac{3}{z}\right| = 2$, then the greatest value of $|z|$ is
(a) 1   (b) 2   (c) 3
(d) 4   (e) 5

103. If the roots of the quadratic equation $mx^2 - nx + k = 0$ are $\tan 33°$ and $\tan 12°$, then the value of $\dfrac{2m + n + k}{m}$ is equal to
(a) 0   (b) 1   (c) 2
(d) 3   (e) 4

104. If α and β are the roots of $4x^2 + 2x - 1 = 0$, then $\beta =$
(a) $-\dfrac{1}{4\alpha}$ (b) $-\dfrac{1}{2\alpha}$ (c) $-\dfrac{1}{\alpha}$

(d) $-\dfrac{1}{3\alpha}$ (e) $\dfrac{1}{\alpha}$

105. If α and α^2 are the roots of the equation
$x^2 - 6x + c = 0$, then the positive value of c is
(a) 2   (b) 3   (c) 4
(d) 9   (e) 8

106. If one of the roots of the quadratic equation
$ax^2 - bx + a = 0$ is 6, then value of $\dfrac{b}{a}$ is equal to
(a) $\dfrac{1}{6}$ (b) $\dfrac{11}{6}$ (c) $\dfrac{37}{6}$
(d) $\dfrac{6}{11}$ (e) $\dfrac{6}{37}$

107. If the equation $2x^2 + (a + 3)x + 8 = 0$ has equal roots, then one of the values of a is
(a) −9   (b) −5   (c) −11
(d) 11   (e) 9

108. If 6th term of a G.P. is 2, then the product of first 11 terms of the G.P. is equal to
(a) 512   (b) 1024   (c) 2048
(d) 256   (e) 32

109. If the product of five consecutive terms of a G.P. is $\dfrac{243}{32}$, then the middle term is
(a) $\dfrac{2}{3}$ (b) $\dfrac{3}{2}$ (c) $\dfrac{4}{3}$
(d) $\dfrac{3}{4}$ (e) 1

110. If a_1, a_2, a_3, a_4 are in A.P., then
$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \frac{1}{\sqrt{a_3} + \sqrt{a_4}} =$$
(a) $\dfrac{\sqrt{a_4} - \sqrt{a_1}}{a_3 - a_2}$ (b) $\dfrac{a_4 - a_1}{a_3 - a_2}$ (c) $\dfrac{a_3 - a_2}{\sqrt{a_4} - \sqrt{a_1}}$
(d) $\dfrac{a_1 - a_4}{a_3 - a_1}$ (e) $\dfrac{a_5 - a_0}{a_1 - a_4}$

111. If $a_1, a_2, a_3, ..., a_{20}$ are in A.P. and $a_1 + a_{20} = 45$, then $a_1 + a_2 + a_3 + ... + a_{20}$ is equal to
(a) 90   (b) 900   (c) 350
(d) 450   (e) 730

112. Sum of the series
$1(1) + 2(1 + 3) + 3(1 + 3 + 5) + 4(1 + 3 + 5 + 7) +$
$.... + 10(1 + 3 + 5 + 7 + ... + 19)$ is equal to
(a) 385   (b) 1025   (c) 1125
(d) 2025   (e) 3025

113. In an A.P., the 6th term is 52 and the 11th term is 112. Then the common difference is equal to
(a) 4   (b) 20   (c) 12
(d) 8   (e) 6

114. If the coefficients of x^3 and x^4 in the expansion of $(3 + kx)^9$ are equal, then the value of k is
(a) 3   (b) $\dfrac{1}{3}$ (c) 2
(d) $\dfrac{1}{2}$ (e) 1

115. The total number of 7 digit positive integral numbers with distinct digits that can be formed using the digits 4, 3, 7, 2, 1, 0, 5 is
(a) 4320   (b) 4340   (c) 4310
(d) 4230   (e) 4220

116. If $^nP_4 = 5\,(^nP_3)$, then the value of n is equal to
(a) 5   (b) 6   (c) 7
(d) 8   (e) 9

117. The remainder when 2^{2016} is divided by 63, is
(a) 1   (b) 8   (c) 17
(d) 32   (e) 61

118. If $^nC_2 + {}^nC_3 = {}^6C_3$ and $^nC_x = {}^nC_3$, $x \neq 3$, then the value of x is equal to
(a) 5   (b) 4   (c) 2
(d) 6   (e) 1

119. If $\displaystyle\sum_{k=0}^{18} \frac{k}{^{18}C_k} = a \sum_{k=0}^{18} \frac{1}{^{18}C_k}$, then the value of a is equal to
(a) 3   (b) 9   (c) 6
(d) 18   (e) 36

120. If the square of the matrix $\begin{pmatrix} a & b \\ a & -a \end{pmatrix}$ is the unit matrix, then b is equal to
(a) $\dfrac{a}{1 + a^2}$ (b) $\dfrac{1 - a^2}{a}$ (c) $\dfrac{1 + a^2}{a}$
(d) $\dfrac{a}{1 - a^2}$ (e) $1 + a^2$

1. (d): We have, $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 1 & 5+5x & 2+x \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$

$\Rightarrow x^2 + 7x + 6 = 0 \Rightarrow (x+6)(x+1) = 0$

$\Rightarrow x = -6$ or $x = -1$

2. (e): We have, $A = \begin{vmatrix} 8 & 27 & 125 \\ 2 & 3 & 5 \\ 1 & 1 & 1 \end{vmatrix}$

Expanding along R_1, we get

$A = 8(3 - 5) - 27(2 - 5) + 125(2 - 3) = -60$

$\therefore A^2 = (-60)^2 = 3600$

3. (a): We have, $A = \begin{bmatrix} x & 1 & -x \\ 0 & 1 & -1 \\ x & 0 & 7 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} x & 1 & -x \\ 0 & 1 & -1 \\ x & 0 & 7 \end{vmatrix}$

$= x(7 - 0) - 1(0 + x) - x(0 - x)$

$\Rightarrow |A| = x^2 + 6x$ ...(i)

Also, $\det(A) = \begin{vmatrix} 3 & 0 & 1 \\ 2 & -1 & 2 \\ 0 & 0 & 3 \end{vmatrix} = 3(-3 - 0) - 0 + 0$

$\therefore \det(A) = -9$ ...(ii)

From (i) & (ii), we have, $x^2 + 6x = -9$

$\Rightarrow x^2 + 6x + 9 = 0$

$\Rightarrow (x + 3)^2 = 0 \Rightarrow x = -3$

4. (a): Let $\Delta = \begin{vmatrix} x^2 & x^3+1 & x^5+2 \\ x^3+3 & x^2+x & x^3+x^4 \\ x+4 & x^3+x^5 & 2^3 \end{vmatrix}$

Expanding along R_1, we get

$\Delta = x^2(8x^2 + 8x - x^6 - x^8 - x^7 - x^9)$

$- (x^3 + 1)(4x^3 + 24 - 5x^4 - x^5) + (x^5 + 2)$

$(x^6 + x^8 + 2x^3 + 3x^5 - 5x^2 - 4x)$

$\therefore$ Coefficient of x^2 in the expansion of Δ

$= -5 \times 2 = -10$

5. (b): We have, $A = \begin{bmatrix} 1 & \omega^2 \\ \omega & 1 \end{bmatrix}$

$\therefore A^2 = \begin{bmatrix} 1 & \omega^2 \\ \omega & 1 \end{bmatrix} \begin{bmatrix} 1 & \omega^2 \\ \omega & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+\omega^3 & 2\omega^2 \\ 2\omega & \omega^3+1 \end{bmatrix} = \begin{bmatrix} 2 & 2\omega^2 \\ 2\omega & 2 \end{bmatrix}$ $(\because \omega^3 = 1)$

$= 2\begin{bmatrix} 1 & \omega^2 \\ \omega & 1 \end{bmatrix} = 2^1 A$

Similarly, $A^3 = A^2 \times A = 2A \times A = 2A^2$

$= 2(2A) = 2^2 A$

$\therefore A^{100} = 2^{99} A$

6. (c): We have, $\dfrac{396}{10} - \dfrac{19-x}{10} < \dfrac{376}{10} - \dfrac{19-9x}{10}$

$\Rightarrow 396 - 19 + x < 376 - 19 + 9x$

$\Rightarrow 20 < 8x \Rightarrow x > \dfrac{5}{2} \Rightarrow x > 2.5$

So, the least integer satisfies the given inequality is 3.

7. (b): Here two cases arises,

Case 1 : When $-\infty < x \le 1$.

Then, $(x - 1) \le 0$, $x - 3 < 0$

$\Rightarrow |x - 1| = -(x - 1)$ and $|x - 3| = -(x - 3)$

Now, $|x - 1| + |x - 3| \le 8$

$\Rightarrow -(x - 1) - (x - 3) \le 8$

$\Rightarrow -2x + 4 \le 8 \Rightarrow x \ge -2$ ...(i)

Case 2 : When $3 \le x < \infty$

$\therefore |x - 1| = (x - 1)$ and $|x - 3| = (x - 3)$

Now, $|x - 1| + |x - 3| \le 8$

$\Rightarrow 2x - 4 \le 8$

$\Rightarrow x \le 6$ ...(ii)

From (i) and (ii), we get

$x \in [-2, 6]$

8. (b): We have, p is false statement, q is true statement and r is also true statement.

Now,

(a) $p \vee (\sim q \wedge r) \equiv F \vee (F \wedge T) \equiv F \vee F \equiv F$

(b) $\sim p \vee (q \wedge r) \equiv T \vee (T \wedge T) \equiv T \vee T \equiv T$

(c) $(p \wedge q) \vee \sim r = (F \wedge T) \vee F \equiv F \vee F \equiv F$

(d) $(p \vee q) \wedge \sim r = (F \vee T) \wedge F \equiv T \wedge F \equiv F$

(e) $\sim p \wedge (\sim q \wedge r) \equiv T \wedge (F \wedge T) \equiv T \wedge F \equiv F$

So, option (b) is the answer.

9. (a): We have, $\sim(p \vee q) \vee \sim(p \vee r)$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ (By De Morgan's Law)

$\equiv \sim p \wedge (\sim q \vee \sim r)$

10. (d)

11. (e) : We have,

$$\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\frac{5\pi}{8} + \sin^2\frac{7\pi}{8}$$

$$= \sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8} + \sin^2\left(\pi - \frac{3\pi}{8}\right) + \sin^2\left(\pi - \frac{\pi}{8}\right)$$

$$= 2\left(\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8}\right)$$

$$= 2\left[\sin^2\frac{\pi}{8} + \sin^2\left(\frac{\pi}{2} - \frac{\pi}{8}\right)\right] = 2\left[\sin^2\frac{\pi}{8} + \cos^2\frac{\pi}{8}\right] = 2$$

12. (a) : We have,

$$\frac{\sqrt{3}}{\sin 15°} - \frac{1}{\cos 15°} = \frac{\sqrt{3}}{\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \frac{1}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}$$

$$= 2\sqrt{2}\left(\frac{3+\sqrt{3}-\sqrt{3}+1}{3-1}\right) = 4\sqrt{2}$$

13. (b) : We have, $\sin x + \cos x = \sqrt{2}$

Squaring both sides, we get

$(\sin^2 x + \cos^2 x) + 2\sin x \cos x = 2$

$\Rightarrow \quad 1 + 2\sin x \cos x = 2$

$\Rightarrow \quad \sin x \cos x = \dfrac{1}{2}$

14. (d) : We have, $\tan\theta = \dfrac{1}{2}$

$$\therefore \quad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{4}{3}$$

Now, $\tan(2\theta + \phi) = \dfrac{\tan 2\theta + \tan\phi}{1 - \tan 2\theta \tan\phi} = \dfrac{\frac{4}{3}+\frac{1}{3}}{1-\frac{4}{3}\times\frac{1}{3}} = 3$

15. (a) : We have, $\tan^{-1} x + \tan^{-1}\left(\dfrac{2}{3}\right) = \tan^{-1}\left(\dfrac{7}{4}\right)$

$$\Rightarrow \quad \tan^{-1} x = \tan^{-1}\left(\frac{7}{4}\right) - \tan^{-1}\left(\frac{2}{3}\right)$$

$$\Rightarrow \quad \tan^{-1} x = \tan^{-1}\left[\frac{\frac{7}{4}-\frac{2}{3}}{1+\frac{7}{4}\times\frac{2}{3}}\right] = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad x = \frac{1}{2}$$

16. (e) : We have, $\tan A - \tan B = x$

$$\Rightarrow \quad \frac{1}{\cot A} - \frac{1}{\cot B} = x \Rightarrow \frac{\cot B - \cot A}{\cot A \cot B} = x$$

$$\Rightarrow \quad \cot A \cot B = \frac{y}{x}$$

Now, $\cot(A - B) = \dfrac{\cot A \cdot \cot B + 1}{\cot B - \cot A}$

$$= \frac{\frac{y}{x}+1}{y} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y}$$

17. (c) : We have, $\tan^{-1} x + \tan^{-1} y = \dfrac{2\pi}{3}$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1} y = \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{3}$$

$$\Rightarrow \quad \left(\frac{\pi}{2} - \tan^{-1} x\right) + \left(\frac{\pi}{2} - \tan^{-1} y\right) = \frac{\pi}{3}$$

$$\Rightarrow \quad \cot^{-1} x + \cot^{-1} y = \frac{\pi}{3}$$

18. (d): If the orthocentre, centroid, circumcentre and incentre of a triangle coincide, then the triangle must be equilateral.
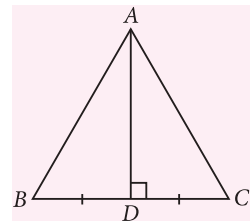
Now, in $\triangle ABC$

$AB = BC = AC = \sqrt{75} = 5\sqrt{3}$

Also, $BD = \dfrac{BC}{2} = \dfrac{5\sqrt{3}}{2}$

In $\triangle BDA$, $AD^2 = AB^2 - BD^2$

$$= 75 - \frac{75}{4} = \frac{225}{4}$$

$$\Rightarrow \quad AD = \frac{15}{2} \text{ units}$$

19. (e) : Let $P(x, y)$ be any point such that $\angle APB = 90°$
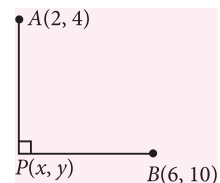
$\Rightarrow$ Slope of $PA \times$ Slope of $PB = -1$

$$\Rightarrow \quad \left(\frac{4-y}{2-x}\right)\left(\frac{10-y}{6-x}\right) = -1$$

$\Rightarrow \quad y^2 - 14y + 40 = -(x^2 - 8x + 12)$

$\Rightarrow \quad x^2 + y^2 - 8x - 14y + 52 = 0$ is the required locus.

20. (c) : Let the fourth vertex be $D(x, y)$

Now, we know that the diagonals of a ||gm bisect each other.

$$\therefore \quad \left(\frac{-2-1}{2}, \frac{1+0}{2}\right) = \left(\frac{-6+x}{2}, \frac{5+y}{2}\right)$$

$$\Rightarrow \quad \left(\frac{-3}{2}, \frac{1}{2}\right) = \left(\frac{x-6}{2}, \frac{y+5}{2}\right)$$

On comparing, we get

$x = 3$ and $y = -4$

So, fourth vertex is $(3, -4)$

21. (c) : The given equation of line can be written as

$2x - 5y = 60 \Rightarrow y = \dfrac{2}{5}x - 12$ ...(i)

On comparing (i) with general equation of line $y = mx + c$, we get slope of line $= \dfrac{2}{5}$

22. (d): The given equation of line can be written in intercept form as, $\dfrac{x}{2} + \dfrac{y}{(-8/a)} = 1$

According to question, $\dfrac{-8}{a} = 2 \times 3 \Rightarrow a = \dfrac{-4}{3}$

23. (a): Equation of line passes through the point $(0, 1)$ is $y - 1 = m(x - 0)$

$\Rightarrow mx - y + 1 = 0$ ...(i)

Now, distance of line (i) from origin is $\dfrac{3}{5}$ units

$\therefore \dfrac{0 + 0 + 1}{\sqrt{m^2 + 1}} = \dfrac{3}{5} \Rightarrow m^2 + 1 = \dfrac{25}{9} \Rightarrow m^2 = \dfrac{16}{9}$

$\Rightarrow m = \pm \dfrac{4}{3}$

So, the equation of lines are

$\dfrac{4}{3}x - y + 1 = 0, \dfrac{-4}{3}x - y + 1 = 0$

$\Rightarrow 4x - 3y + 3 = 0, 4x + 3y - 3 = 0$

24. (d): Let (α, β) be any point on the line $x + y - 3 = 0$.

$\therefore \alpha + \beta = 3$ ...(i)

Also, the line passing through (α, β) and $(3, -2)$ is nearest to the given line if both lines are perpendicular to each other *i.e.*, $m_1 m_2 = -1$

$\therefore -1 \times \dfrac{\beta + 2}{\alpha - 3} = -1$

$\Rightarrow \alpha - 3 = \beta + 2 \Rightarrow \alpha - \beta = 5$ ...(ii)

On solving (i) and (ii), we get

$\alpha = 4, \beta = -1$

25. (e): Let $M(h, k)$ be any point on the line $4x + 3y = 25$.
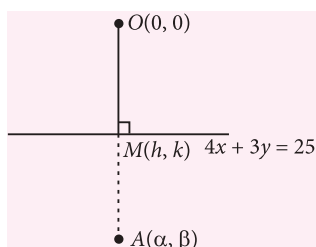
Then, (slope of OM) $\times$ (slope of $4x + 3y = 25$) $= -1$

$\Rightarrow \dfrac{k}{h} \times \dfrac{-4}{3} = -1 \Rightarrow k = \dfrac{3}{4}h$ ...(i)

Since $M(h, k)$ lies on $4x + 3y = 25$

$\therefore 4h + 3k = 25$ ...(ii)

On solving (i) and (ii), we get $h = 4, k = 3$

Also, let $A(\alpha, \beta)$ be the image of the origin with respect to the line $4x + 3y = 25$.

Then, $h = \dfrac{\alpha + 0}{2}, k = \dfrac{\beta + 0}{2} \Rightarrow \alpha = 8, \beta = 6$

So, the image of $O(0, 0)$ is $(8, 6)$

26. (d): Radius $= \sqrt{\left(\dfrac{g}{a}\right)^2 + \left(\dfrac{f}{a}\right)^2 - \dfrac{c}{a}}$

$\Rightarrow r = \sqrt{\left(\dfrac{4}{4}\right)^2 + \left(\dfrac{-8}{4}\right)^2 - \dfrac{\lambda}{4}} \Rightarrow r^2 = \dfrac{20 - \lambda}{4}$

Now, Area $= \pi r^2$

$\Rightarrow 9\pi = \pi\left(\dfrac{20 - \lambda}{4}\right) \Rightarrow \lambda = -16$

27. (c): Let the equation of circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

Since (i) passes through $(2, 3)$, $(2, 7)$ and $(5, 3)$

$\therefore 2^2 + 3^2 + 2(2)g + 2(3)f + c = 0$

$\Rightarrow 4g + 6f + c = -13$ ...(ii)

Similarly, $4g + 14f + c = -53$ ...(iii)

and $10g + 6f + c = -34$ ...(iv)

On solving (ii), (iii) and (iv), we get

$g = \dfrac{-7}{2}, f = -5, c = 31$

$\therefore$ Radius $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{\left(\dfrac{-7}{2}\right)^2 + (-5)^2 - 31} = \dfrac{5}{2}$ units

28. (c): Centre of the given circle is $(1, 3)$

$\therefore$ Distance between the centres of the circles
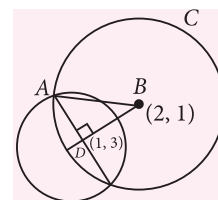
$BD = \sqrt{(2 - 1)^2 + (1 - 3)^2} = \sqrt{5}$ units

Also, AD = radius of the given circle

$= \sqrt{(-1)^2 + (-3)^2 - 6} = 2$ units

In right ΔABD,

$AB = \sqrt{BD^2 + AD^2} = 3$ units

So, radius of circle $C = 3$ units

29. (a): We have, $x^2 + y^2 = \dfrac{k}{2}$ ...(i)

Since (i) passes through $(1, 1)$

$\Rightarrow 1 + 1 = \dfrac{k}{2} \Rightarrow k = 4$

$\therefore$ (i) becomes, $x^2 + y^2 = 2 = (\sqrt{2})^2$

So, radius of circle $= \sqrt{2}$ units

30. (e): Let S be the focus and M be any point on the directrix of the ellipse.

Then, $\dfrac{PS^2}{PM^2} = e^2 \Rightarrow \dfrac{(8)^2}{PM^2} = \left(\dfrac{4}{5}\right)^2$

$\Rightarrow PM = 10$ units

31. (b): Let $(a, 0)$ be the coordinates of focus.

Now, $\dfrac{a+0}{2} = -3 \Rightarrow a = -6$

So, focus $\equiv (-6, 0)$

32. (d): The given equation of ellipse can be written as,

$\dfrac{x^2}{\left(\dfrac{1}{4}\right)} + \dfrac{y^2}{\left(\dfrac{1}{9}\right)} = 1$

Here $a > b$

$\therefore \quad e = \sqrt{1 - \dfrac{b^2}{a^2}} = \sqrt{1 - \dfrac{4}{9}} = \dfrac{\sqrt{5}}{3}$

$\therefore \quad$ Coordinates of foci $= (\pm\, ae, 0) = \left(\pm\dfrac{\sqrt{5}}{6}, 0\right)$

33. (e): Distance of focus from directrix $= 2a$

$\Rightarrow \dfrac{4+8}{\sqrt{1^2}} = 2a \Rightarrow a = 6$

So, length of latus rectum $= 4a = 24$ units

34. (b): The given equation of ellipse can be written as,

$\dfrac{x^2}{4} + \dfrac{y^2}{a} = 1 \ (a < 4)$

$\therefore \quad e = \sqrt{1 - \dfrac{b^2}{a^2}} \Rightarrow \dfrac{1}{\sqrt{2}} = \sqrt{1 - \dfrac{a}{4}}$

$\Rightarrow \dfrac{4-a}{4} = \dfrac{1}{2} \Rightarrow a = 2$

$\therefore \quad b^2 = 2$

So, length of semi-minor axis $= b = \sqrt{2}$ units

35. (d): The given equation of hyperbola is

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ ...(i)

Now (i) passes through $(\sqrt{6}, 3)$

$\therefore \quad \dfrac{6}{a^2} - \dfrac{9}{b^2} = 1$ ...(ii)

Also, length of latus rectum $= \dfrac{2b^2}{a}$

$\Rightarrow \dfrac{18}{5} = \dfrac{2b^2}{a}$ ...(iii)

On solving (ii) and (iii), we get $a = -6, 1$
So, length of transverse axis $= 2a = 2$

36. (e): We have, projection of $\vec{a}$ on $\vec{b} = \dfrac{-9}{\sqrt{3}}$

$\Rightarrow \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{-9}{\sqrt{3}} \Rightarrow |\vec{a}|\cdot\cos\left(\dfrac{5\pi}{6}\right) = \dfrac{-9}{\sqrt{3}}$

$\Rightarrow |\vec{a}| = 6$

37. (d)

38. (a): It is given that $\vec{a}$ and $\vec{b}$ are collinear.

$\therefore \quad \dfrac{2}{4/7} = \dfrac{-1}{(-2/7)} = \dfrac{-m}{2} \Rightarrow m = -7$

39. (b): $(3\vec{a} - 5\vec{b}) = (6\hat{i} + 15\hat{j} - 21\hat{k}) - (5\hat{i} + 15\hat{j} + 25\hat{k})$

$\qquad\qquad = \hat{i} - 46\hat{k}$

$4\vec{a} \times 5\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 20 & -28 \\ 5 & 15 & 25 \end{vmatrix} = 920\hat{i} - 340\hat{j} + 20\hat{k}$

$\therefore \quad (3\vec{a} - 5\vec{b})\cdot(4\vec{a} \times 5\vec{b}) = 920 - 920 = 0$

40. (d): We have, $\vec{a} + 2\vec{b} = \vec{c}$ ...(i)

Taking cross product on both sides of (i) with $\vec{a}$, we get

$2\vec{b} \times \vec{a} = \vec{c} \times \vec{a} \Rightarrow \vec{c} \times \vec{a} = -2(\vec{a} \times \vec{b})$ ...(ii)

Again taking cross product on both sides of (i) with $\vec{c}$, we get

$\vec{a} \times \vec{c} + 2\vec{b} \times \vec{c} = \vec{0} \Rightarrow \vec{b} \times \vec{c} = \dfrac{1}{2}(\vec{c} \times \vec{a}) = -\vec{a} \times \vec{b}$ ...(iii)

[using (ii)]

Now, $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \lambda\, \vec{a} \times \vec{b}$

$\Rightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{b} - 2\vec{a} \times \vec{b} = \lambda\, \vec{a} \times \vec{b}$ [Using (ii) & (iii)]

$\Rightarrow -2\vec{a} \times \vec{b} = \lambda\, \vec{a} \times \vec{b} \Rightarrow \lambda = -2$

41. (e): $(\vec{a} + \vec{b})\cdot\vec{b} = |\vec{a} + \vec{b}||\vec{b}|\cos 60°$

$\Rightarrow \cos 60° = \dfrac{|\vec{b}|}{|\vec{b} + \vec{a}|} \Rightarrow 2|\vec{b}| = |\vec{a} + \vec{b}|$

$\Rightarrow |\vec{a} + \vec{b}|^2 = 4|\vec{b}|^2 \Rightarrow |\vec{a}|^2 + |\vec{b}|^2 = 4|\vec{b}|^2$

$\Rightarrow |\vec{a}|^2 = 3|\vec{b}|^2 \Rightarrow |\vec{a}| = \sqrt{3}|\vec{b}|$

42. (d): Since $\vec{a} - \vec{b}$ and $\vec{a} + \vec{b}$ are perpendicular

$\therefore \quad (\vec{a} - \vec{b})\cdot(\vec{a} + \vec{b}) = 0 \Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 0$

$\Rightarrow |\vec{a}|^2 = \left(\sqrt{(3)^2 + (-4)^2 + 2^2}\right)^2 = 29$

$\Rightarrow |\vec{a}| = \sqrt{29}$

43. (e): The given line can be written as,
$\vec{r} = (1 + 2\alpha)\hat{i} + (1 - \alpha)\hat{j} + (1 + 4\alpha)\hat{k}$ ...(i)
Since (i) passes through xy plane.
$\therefore \quad 1 + 4\alpha = 0 \implies \alpha = \dfrac{-1}{4}$

So, $x = 1 + 2\alpha = 1 + 2 \times \left(\dfrac{-1}{4}\right) = \dfrac{1}{2}$

$y = 1 - \alpha = 1 + \dfrac{1}{4} = \dfrac{5}{4}$

$\therefore$ The coordinates of required point are $\left(\dfrac{1}{2}, \dfrac{5}{4}, 0\right)$

44. (b): Equation of plane parallel to
$2x - 5y + 7z + 11 = 0$ is $2x - 5y + 7z + k = 0$ ...(i)
Since plane (i) passes through $(-1, 5, -7)$
$\therefore \quad 2(-1) - 5(5) + 7(-7) + k = 0 \implies k = 76$
So, required equation of plane is
$2x - 5y + 7z + 76 = 0$ i.e., $\vec{r} \cdot (2\hat{i} - 5\hat{j} + 7\hat{k}) + 76 = 0$

45. (a): Let $A \equiv (1, 2, 3) \therefore \overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$,
$\overrightarrow{OP} = 2\hat{i} + 4\hat{j} + 5\hat{k}, \overrightarrow{OQ} = 3\hat{i} + 3\hat{j} + \hat{k}$
$\therefore \quad \overrightarrow{AP} = \hat{i} + 2\hat{j} + 2\hat{k}, \overrightarrow{AQ} = 2\hat{i} + \hat{j} - 2\hat{k}$
Now, $\overrightarrow{AP} \cdot \overrightarrow{AQ} = |\overrightarrow{AP}||\overrightarrow{AQ}| \cos\theta$, where θ is the angle
between $\overrightarrow{AP}$ and $\overrightarrow{AQ}$
$\implies 2 + 2 - 4 = \sqrt{1 + 4 + 4} \times \sqrt{4 + 1 + 4} \cos\theta$
$\implies \cos\theta = 0 \implies \theta = 90°$

46. (c): The given lines are
$\dfrac{x - 1}{2} = \dfrac{y - 1}{a} = \dfrac{z - 0}{4}$ ...(i)

and $\dfrac{x - 3}{1} = \dfrac{y - \dfrac{3}{2}}{2} = \dfrac{z - 2}{2}$ ...(ii)

$\because$ Lines (i) & (ii) are perpendicular.
$\therefore \quad (2)(1) + (a)(2) + (4)(2) = 0$
$\implies 2a + 10 = 0 \implies a = -5$

47. (a): Given line is,
$\dfrac{x + 1}{2} = \dfrac{y + 1}{3} = \dfrac{z + 1}{4} = r$ (say)
Any point P on this line is $(2r - 1, 3r - 1, 4r - 1)$.
This point lies on the plane $x + 2y + 3z = 14$
$\therefore \quad (2r - 1) + 2(3r - 1) + 3(4r - 1) = 14$
$\implies 20r = 20 \implies r = 1$
So, coordinates of P are $(1, 2, 3)$
$\therefore$ Required distance $= \sqrt{(1 - 0)^2 + (2 - 0)^2 + (3 - 0)^2}$
$= \sqrt{14}$ units

48. (e): Given lines can be written as
$\dfrac{x - 3}{-1} = \dfrac{y + 4}{-2} = \dfrac{z - 5}{2} = \lambda$ ...(i)

and $\dfrac{x - 3}{1} = \dfrac{y + 4}{2} = \dfrac{z - 5}{7}$ ...(ii)

Any point on line (i) is,
$P \equiv (-\lambda + 3, -2\lambda - 4, 2\lambda + 5)$
Since both the lines intersect each other. Therefore, P
lies on line (ii).
$\implies \dfrac{-\lambda + 3 - 3}{1} = \dfrac{-2\lambda - 4 + 4}{2} = \dfrac{2\lambda + 5 - 5}{7}$
$\implies \lambda = 0$
$\therefore$ Required point is $(3, -4, 5)$

49. (e): We have, $\dfrac{x - 2}{1} = \dfrac{y}{-3} = \dfrac{z - 1}{-2} = t$ (say)

$\implies x = t + 2, y = -3t, z = -2t + 1$
So, the vector equation of the line is,
$x\hat{i} + y\hat{j} + z\hat{k} = (t + 2)\hat{i} - 3t\hat{j} + (-2t + 1)\hat{k}$
$\implies \vec{r} = (2\hat{i} + \hat{k}) + t(\hat{i} - 3\hat{j} - 2\hat{k})$

50. (b): Point on the given line is $(1, 1, 2)$.
Now, we have to find the distance between the point
$(1, 1, 2)$ and the plane $2x + y - 3z = 5$
$\therefore$ Required distance $= \dfrac{|2 + 1 - 6 - 5|}{\sqrt{2^2 + 1^2 + (-3)^2}} = \dfrac{8}{\sqrt{14}}$ units

51. (a): Let S be the sample space and A be the event
of getting the sum as a composite number.
$\therefore \quad A = \{(1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1),$
$(3, 3), (3, 5), (3, 6), (4, 2), (4, 4), (4, 5), (4, 6), (5, 1),$
$(5, 3), (5, 4), (5, 5), (6, 2), (6, 3), (6, 4), (6, 6)\}$
So, required probability $= \dfrac{n(A)}{n(S)} = \dfrac{21}{36} = \dfrac{7}{12}$

52. (c): According to question,
Mean $= 6 \implies a + b + 8 + 5 + 10 = 30$
$\implies a + b = 7$ ...(i)
Also, variance $= 6.8$
$\implies (a - 6)^2 + (b - 6)^2 + 2^2 + 1^2 + 4^2 = (6.8) \times 5$
$\implies a^2 + b^2 = 25$ ...(ii)
On solving (i) and (ii), we get $ab = 12$

53. (d): Let the total number of students be x.
Then according to question,
Average of remaining $(x - 12)$ students, $= 72$
$\therefore \quad \dfrac{10(100) + 2(0) + 72(x - 12)}{x} = 76$
$\implies 1000 + 72x - 864 = 76x \implies x = 34$

54. (a) : Required probability = 1 – probability of getting 2 balls of same colour

$$= 1 - \left(\frac{{}^3C_2 + {}^4C_2 + {}^5C_2}{{}^{12}C_2} \right) = 1 - \frac{19}{66} = \frac{47}{66}$$

55. (e) : Required probability $= \dfrac{{}^5C_2 \times {}^6C_1 + {}^6C_3}{{}^{11}C_3} = \dfrac{16}{33}$

56. (b) : We have, $\lim\limits_{x \to 0} \dfrac{\cot 4x}{\operatorname{cosec} 3x} = \lim\limits_{x \to 0} \dfrac{\sin 3x}{\tan 4x}$

$$= 3 \lim_{x \to 0} \left(\frac{\sin 3x}{3x} \right) \times \frac{1}{4} \lim_{x \to 0} \left(\frac{4x}{\tan 4x} \right) = \frac{3}{4}$$

57. (e) : We have, R.H.L. $= \lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} \cos h = 1$

L.H.L. $= \lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} (-\cos h) = -1$

Since L.H.L. $\neq$ R.H.L.

∴ $f(x)$ is not continuous at $x = 0$

58. (a) : We have, $\lim\limits_{n \to \infty} \left(\dfrac{{}^nC_3 - {}^nP_3}{n^3} \right)$

$$= \lim_{n \to \infty} \left(\frac{(-5)n(n-1)(n-2)}{6n^3} \right)$$

$$= \frac{-5}{6} \lim_{n \to \infty} \left(\left(1 - \frac{1}{n} \right) \left(1 - \frac{2}{n} \right) \right) = \frac{-5}{6}$$

59. (b) : We have, $f(x) = 3x + 5$, $g(x) = x^2 - 1$

∴ $(fog)(x^2 - 1) = f[g(x^2 - 1)] = f[(x^2 -1)^2 -1]$
$= f(x^4 - 2x^2) = 3(x^4 - 2x^2) + 5$
$= 3x^4 - 6x^2 + 5$

60. (d) : Let p be the period.

∴ $f(x + p) = f(x)$
$\Rightarrow$ $\tan(4(x + p) - 1) = \tan(4x - 1)$
$\Rightarrow$ $4(x + p) -1 = n\pi + (4x - 1)$, $n \in Z$
$\Rightarrow$ $4p = n\pi$, $n \in Z$
$\Rightarrow$ $p = \dfrac{n\pi}{4}$, $n \in Z$

61. (b) : We have, $2^x + 2^y = 2^{x+y}$

Differentiating w.r.t. x, we get

$$2^x \cdot \log 2 + 2^y \cdot \log 2 \cdot \frac{dy}{dx} = 2^{x+y} \cdot \log 2 \left(1 + \frac{dy}{dx} \right)$$

$$\Rightarrow \frac{dy}{dx}(2^y - 2^{x+y}) = 2^{x+y} - 2^x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2^{x+y} - 2^x}{2^y - 2^{x+y}}$$

$$\therefore \quad \frac{dy}{dx}\bigg|_{\text{at }(1,1)} = \frac{2^2 - 2}{2 - 2^2} = -1$$

62. (b) : We have, $f(x) = \dfrac{\sin^{-1} x}{\sqrt{1 - x^2}}$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{\sqrt{1 - x^2} \times \dfrac{1}{\sqrt{1 - x^2}} - \sin^{-1} x \times \dfrac{1}{2\sqrt{1 - x^2}} \times (-2x)}{1 - x^2}$$

$$\Rightarrow (1 - x^2)f'(x) = 1 + x \left(\frac{\sin^{-1} x}{\sqrt{1 - x^2}} \right)$$

$$\Rightarrow (1 - x^2) f'(x) - x f(x) = 1$$

63. (a) : We have, $f(x) = \dfrac{x^{10}}{2^{10}} = \dfrac{1}{2^{10}} \times x^{10}$

$$\therefore \quad f'(x) = \frac{1}{2^{10}} \times 10x^9 \Rightarrow f'(1) = \frac{10}{2^{10}}$$

$$f''(x) = \frac{1}{2^{10}} \times 10 \times 9 \times x^8 \Rightarrow f''(1) = \frac{10 \times 9}{2^{10}}$$

and so on.

$$\therefore \quad f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \dots + \frac{f^{10}(1)}{10!}$$

$$= \frac{1}{2^{10}} + \frac{10}{2^{10} \times 1!} + \dots + \frac{10!}{10! \times 2^{10}}$$

$$= \frac{1}{2^{10}} \left[1 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + \dots + {}^{10}C_{10} \right]$$

$$= \frac{1}{2^{10}} \times 2^{10} = 1$$

64. (d): $\left(\dfrac{f}{g} \right)'(x) = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

$$\therefore \left(\frac{f}{g} \right)'(4) = \frac{g(4) \cdot f'(4) - f(4) \cdot g'(4)}{[g(4)]^2}$$

$$= \frac{6 \times 5 - \dfrac{1}{3} \times 12}{36} = \frac{13}{18}$$

65. (e) : Let $y = (ax - 5)e^{3x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = (ax - 5) \cdot 3 \cdot e^{3x} + e^{3x}(a)$$

$$\frac{dy}{dx}\bigg|_{\text{at }x=0} = -15 + a \Rightarrow -13 = -15 + a \Rightarrow a = 2$$

66. (b) : We have, $y = \tan^{-1}(\sec x + \tan x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+(\sec x + \tan x)^2}[\sec x \cdot \tan x + \sec^2 x]$$

$$= \frac{1}{2\sec^2 x + 2\sec x \tan x} \times \sec x (\tan x + \sec x)$$

$$= \frac{1}{2\sec x(\sec x + \tan x)} \times \sec x (\sec x + \tan x) = \frac{1}{2}$$

67. (d): We have, $s = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right) = \cos^{-1}(2x^2-1)$

$\Rightarrow \quad s = 2\cos^{-1} x$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{ds}{dx} = 2 \times \frac{-1}{\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

Also, $t = \sqrt{1-x^2}$ ...(ii)

Differentiating (ii) w.r.t. x, we get

$$\frac{dt}{dx} = \frac{1 \times (-2x)}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

So, $\dfrac{ds}{dt} = \dfrac{ds}{dx} \times \dfrac{dx}{dt} = \dfrac{2}{x}$

$\therefore \quad \left.\dfrac{ds}{dt}\right|_{\text{at } x=\frac{1}{2}} = 4$

68. (e): Let $y = 2x^3 - 9x^2 + 12x + 4$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$$

For maximum / minimum, $\dfrac{dy}{dx} = 0$

$\Rightarrow \quad 6(x-1)(x-2) = 0$
$\Rightarrow \quad x = 1$ or $x = 2$

Now, $\dfrac{d^2y}{dx^2} = 12x - 18$

$\left.\dfrac{d^2y}{dx^2}\right|_{\text{at } x=1} = 12 - 18 = -6 < 0$

$\left.\dfrac{d^2y}{dx^2}\right|_{\text{at } x=2} = 24 - 18 = 6 > 0$

So, $f(x)$ is minimum at $x = 2$.
And minimum value of $f(x)$
$= 2(2)^3 - 9(2)^2 + 12(2) + 4 = 8$

69. (d): We have, $y = e^x \cos x$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -e^x \sin x + \cos x \cdot e^x = e^x(\cos x - \sin x)$$

$\therefore$ Slope $(S) = \dfrac{dy}{dx} = e^x(\cos x - \sin x)$

Differentiating S w.r.t. x, we get

$$\frac{dS}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = e^x[-\sin x - \cos x + \cos x - \sin x]$$

$$= e^x(-2\sin x)$$

For maximum / minimum, $\dfrac{dS}{dx} = 0$

$\Rightarrow \quad \sin x = 0 \Rightarrow x = 0$ $[\because \ x \in (-\pi, \pi)]$

Now, $\dfrac{d^2S}{dx^2} = -2e^x(\cos x + \sin x) < 0$

$\therefore$ Slope is maximum at $x = 0$

70. (c): By Lagrange's Mean Value Theorem, we have,

$$f'(c) = \frac{f(b) - f(a)}{b-a} = \frac{f(6) - f(0)}{6} = 3, c \in (0, 6)$$

$\therefore$ For some point between $x = 0$ and $x = 6$, $f'(x) = 3$

71. (a): We have, $y = x^3 - 6x + 5$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 6$$

$\therefore$ Slope $= \left.\dfrac{dy}{dx}\right|_{\text{at}(2,1)} = 12 - 6 = 6$

So, required equation of tangent is
$y - 1 = 6(x - 2) \Rightarrow 6x - y - 11 = 0$

72. (b): We have, $y = f(x) = 2x^3 - 5x^2 - 4x + 3$
Slope of curve $(m_1) = 6x^2 - 10x - 4$
Since tangent to $f(x)$ is parallel to x-axis.
$\therefore \quad m_1 = 0 \Rightarrow 6x^2 - 10x - 4 = 0$

$\Rightarrow \quad (3x + 1)(x - 2) = 0 \Rightarrow x = \dfrac{-1}{3}$ or $x = 2$

When $x = 2$, $y = 2(2)^3 - 5(2)^2 - 4(2) + 3 = -9$
So, required point is $(2, -9)$

73. (b): Let θ be the angle between the given sides.

Then, $A = \dfrac{1}{2} \times 8 \times 5 \times \sin\theta$

$\Rightarrow \quad A = 20\sin\theta$ ...(i)

Differentiating (i) w.r.t. t, we get

$$\frac{dA}{dt} = 20\cos\theta \frac{d\theta}{dt} = 20\cos\frac{\pi}{3} \times 0.08 = 0.8 \ \text{m}^2/\text{sec}$$

74. (e): We have, $y = 8x^3 - 60x^2 + 144x + 27$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = 24x^2 - 120x + 144 = 24(x^2 - 5x + 6)$

$= 24(x - 3)(x - 2)$

For strictly decreasing, $\frac{dy}{dx} < 0$

$\Rightarrow (x - 3)(x - 2) < 0 \Rightarrow x \in (2, 3)$

75. (c): Let $I = \int (\sec x)^m \cdot \tan x (\tan^2 x + 1) dx$

$= \int (\sec x)^m \sec^2 x \cdot \tan x \, dx$

Put $\sec x = t \Rightarrow \sec x \cdot \tan x \, dx = dt$

$\therefore \quad I = \int (t)^{m+1} dt = \frac{t^{m+2}}{m+2} + C$

$\Rightarrow I = \frac{(\sec x)^{m+2}}{m+2} + C$

76. (c): Let $I = \int \frac{1}{7} \sin\left(\frac{x}{7} + 10\right) dx$

$= \frac{-1}{7} \cos\left(\frac{x}{7} + 10\right) \times 7 + C$

$= -\cos\left(\frac{x}{7} + 10\right) + C$

77. (b): Let $I = \int \left(\frac{x-a}{x} - \frac{x}{x+a}\right) dx$

$= -a \int \frac{a}{x(x+a)} dx = -a \int \left(\frac{1}{x} - \frac{1}{x+a}\right) dx$

$= -a\log|x| + a\log|x+a| + C$

$= a\log\left|\frac{x+a}{x}\right| + C$

78. (c): Let $I = \int x^4 \cdot e^{x^5} \cdot \cos\left(e^{x^5}\right) dx$

Put $e^{x^5} = t \Rightarrow 5x^4 \cdot e^{x^5} dx = dt$

$\therefore \quad I = \frac{1}{5} \int \cos t \, dt = \frac{1}{5} \sin t + C$

$\Rightarrow I = \frac{1}{5} \sin\left(e^{x^5}\right) + C$

79. (c): Let $I = \int \left(\frac{2x + \sin 2x}{1 + \cos 2x}\right) dx$

$= \int \left(\frac{2x + 2\sin x \cos x}{2\cos^2 x}\right) dx$

$\Rightarrow I = \int x \sec^2 x \, dx + \int \tan x \, dx$

$= x \cdot \tan x - \int \tan x \, dx + \int \tan x \, dx$

$= x \tan x + C$

80. (a): Let $I = \int \frac{1}{\sin x \cos x} dx = \int \frac{2}{\sin 2x} dx$

$= 2 \int \operatorname{cosec} 2x \, dx = \frac{2\log|\tan x|}{2} + C$

$= \log|\tan x| + C$

81. (c): Let $I = \int \frac{1}{8\sin^2 x + 1} dx = \int \frac{\sec^2 x}{\sec^2 x + 8\tan^2 x} dx$

$= \int \frac{\sec^2 x \, dx}{1 + 9\tan^2 x} = \int \frac{\sec^2 x}{1 + (3\tan x)^2} dx$

Put $3\tan x = t \Rightarrow 3\sec^2 x \, dx = dt$

$\therefore \quad I = \frac{1}{3} \int \frac{dt}{1 + t^2} = \frac{1}{3} \tan^{-1}|t| + C$

$= \frac{1}{3} \tan^{-1}(3\tan x) + C$

82. (e): Let $I = \int\limits_0^{\pi/2} \log\left(\frac{\cos x}{\sin x}\right) dx$

$\Rightarrow I = \int\limits_0^{\pi/2} \log(\cot x) dx \qquad \text{...(i)}$

$\Rightarrow I = \int\limits_0^{\pi/2} \log\left(\cot\left(\frac{\pi}{2} - x\right)\right) dx$

$\Rightarrow I = \int\limits_0^{\pi/2} \log(\tan x) dx \qquad \text{...(ii)}$

Adding (i) and (ii), we get

$2I = \int\limits_0^{\pi/2} \log(\cot x) + \log(\tan x) dx = \int\limits_0^{\pi/2} \log(1) dx = 0$

$\therefore \quad I = 0$
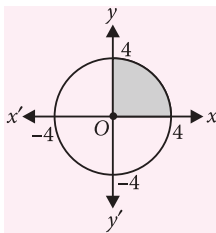
83. (a): Let $I = \int\limits_{-1}^{2} 4x^2 |x| \, dx$

$= \int\limits_{-1}^{0} -4x^3 dx + \int\limits_0^2 4x^3 dx = \left[-4\frac{x^4}{4}\right]_{-1}^0 + \left[4\frac{x^4}{4}\right]_0^2$

$= 1 + 16 = 17$

84. (d): Required area $= \int\limits_0^4 \sqrt{16 - x^2} \, dx$

$$= \left[\frac{x}{2}\sqrt{16 - x^2} + 8\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4$$

$$= 0 + 8 \times \frac{\pi}{2} - 0 - 0$$

$= 4\pi$ sq. units

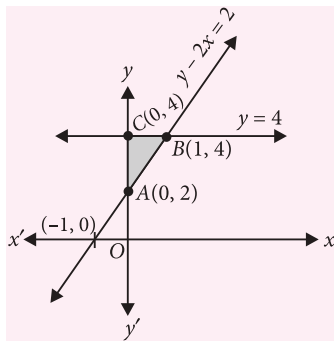**85. (e) :** Let $I = \int\limits_2^4 (x-2)(x-3)(x-4)\,dx$

$$= \int\limits_2^4 (x^3 - 9x^2 + 26x - 24)\,dx$$

$$= \left[\frac{x^4}{4} - 3x^3 + 13x^2 - 24x\right]_2^4 = 0$$

86. (a) : Required area $= \int\limits_2^4 \left(\frac{y-2}{2}\right)dy$

$$= \frac{1}{2}\left[\frac{y^2}{2}\right]_2^4 - [y]_2^4 = 1 \text{ sq. unit}$$

87. (b) : We have, $(x + y + 3)\dfrac{dy}{dx} = 1$ ...(i)

Put $x + y + 3 = t \Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\therefore$ Equation (i) becomes, $t\left(\dfrac{dt}{dx} - 1\right) = 1$

$\Rightarrow \dfrac{dt}{dx} = \dfrac{1}{t} + 1 \Rightarrow \dfrac{t}{t+1}dt = dx$

Integrating both sides, we get

$$\int\left(1 - \frac{1}{t+1}\right)dt = \int dx \Rightarrow t - \log(t+1) = x + C_1$$

$\Rightarrow (x + y + 3) - \log(x + y + 4) = x + C_1$

$\Rightarrow \log(x + y + 4) = y + C_2$, where $C_2 = 3 - C_1$

$\Rightarrow x + y + 4 = Ce^y$, where $C = e^{C_2}$

88. (e) : Given family of curves is

$y^2 = a^2 x + ab$ ...(i)

Differentiating (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = a^2 \Rightarrow y\frac{dy}{dx} = \frac{a^2}{2}$$ ...(ii)

Again differentiating (ii) w.r.t. x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$\therefore$ Order = 2, degree = 1

89. (a) : We have, $\dfrac{x\dfrac{dy}{dx} - y}{\sqrt{x^2 - y^2}} = 10x^2$

$$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = 10x \cdot \sqrt{1 - \frac{y^2}{x^2}}\,dx$$ ...(i)

Put $\dfrac{y}{x} = t \Rightarrow \dfrac{x\,dy - y\,dx}{x^2} = dt$

$\therefore$ Equation (i) becomes, $\dfrac{dt}{\sqrt{1 - t^2}} = 10x\,dx$

Integrating both sides, we get

$$\int \frac{dt}{\sqrt{1 - t^2}} = 10\int x\,dx$$

$\Rightarrow \sin^{-1}t = \dfrac{10x^2}{2} + C \Rightarrow \sin^{-1}\left(\dfrac{y}{x}\right) = 5x^2 + C$

90. (a) : We have, $x\,dy - y\,dx = y^2\,dx$

$$\Rightarrow \frac{y\,dx - x\,dy}{y^2} = -dx$$

Integrating both sides, we get

$$\int\frac{y\,dx - x\,dy}{y^2} = -\int dx$$

$\Rightarrow \dfrac{x}{y} = -x + C \Rightarrow y = \dfrac{x}{C - x}$

91. (d): We have, $a * b = a^b$

$\therefore (2 * 3) * 2 = (2^3) * 2 = 8 * 2 = 8^2 = 64$

92. (c)

93. (e) : We have, $g(f(a)) = f(g(a))$

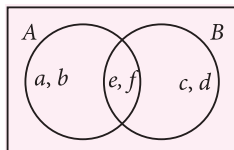$\Rightarrow g(a^3) = f(3^a) \Rightarrow 3^{a^3} = (3^a)^3 \Rightarrow 3^{a^3} = 3^{3a}$

On comparing, we get

$a^3 = 3a \Rightarrow a(a^2 - 3) = 0 \Rightarrow a = 0, \pm\sqrt{3}$

94. (d): We have, $f\left(\dfrac{x+1}{2x-1}\right) = 2x$

Put $x = 1$ on both sides, we get $f(2) = 2$

95. (b): We have, $A\backslash B = \{a, b\}$ i.e., $A - B = \{a, b\}$

$B\backslash A = \{c, d\}$ i.e., $B - A = \{c, d\}$
and $A \cap B = \{e, f\}$.
So, from the venn diagram,
we get, $B = \{c, d, e, f\}$

96. (d)

97. (b): We have, $z = \dfrac{1 + \dfrac{\sqrt{3}}{2} + \dfrac{i}{2}}{1 + \dfrac{\sqrt{3}}{2} - \dfrac{i}{2}}$

$\Rightarrow z = \dfrac{2+\sqrt{3}+i}{2+\sqrt{3}-i} \times \dfrac{2+\sqrt{3}+i}{2+\sqrt{3}+i} = \dfrac{(3+2\sqrt{3})+i(2+\sqrt{3})}{4+2\sqrt{3}}$

$\therefore \arg(z) = \tan^{-1}\left(\dfrac{2+\sqrt{3}}{3+2\sqrt{3}}\right) = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

98. (e): We have, $\dfrac{(1+i)(2+3i)(3-4i)}{(2-3i)(1-i)(3+4i)} = a + ib$

Taking modulus on both sides, we get

$\left|\dfrac{(1+i)(2+3i)(3-4i)}{(2-3i)(1-i)(3+4i)}\right| = |a+ib|$

$\Rightarrow \dfrac{\sqrt{1+1} \times \sqrt{4+9} \times \sqrt{9+16}}{\sqrt{4+9} \times \sqrt{1+1} \times \sqrt{9+16}} = \sqrt{a^2 + b^2}$

$\Rightarrow a^2 + b^2 = 1$

99. (c): We have, $\arg(zw) = \pi$
$\Rightarrow \arg(z) + \arg(w) = \pi$ ...(i)
Also, $\overline{z} + iw = 0 \Rightarrow \overline{z} = i\overline{w}$
$\Rightarrow \arg(\overline{z}) = \arg(i) + \arg(\overline{w})$

$\Rightarrow \arg(z) - \arg(w) = \dfrac{-\pi}{2}$ ...(ii)

On solving (i) and (ii), we get $\arg(z) = \dfrac{\pi}{4}$

100. (a): We have, $z = \dfrac{2-i}{i}$

$\Rightarrow z^2 = \dfrac{(2-i)^2}{i^2} = -3 + 4i$

$\therefore \quad \text{Re}(z^2) + \text{Im}(z^2) = -3 + 4 = 1$

101. (c): Let $z = x + iy$
Then, $|z + 1| < |z - 1| \Rightarrow |x + iy + 1| < |x + iy - 1|$

$\Rightarrow \sqrt{(x+1)^2 + y^2} < \sqrt{(x-1)^2 + y^2}$

$\Rightarrow x^2 + 1 + 2x < x^2 + 1 - 2x \Rightarrow 4x < 0 \Rightarrow x < 0$

102. (c): We have, $\left|z - \dfrac{3}{z}\right| = 2$

Now, $|z| = \left|\left(z - \dfrac{3}{z}\right) + \dfrac{3}{z}\right|$

$\qquad \leq \left|z - \dfrac{3}{z}\right| + \left|\dfrac{3}{z}\right| = 2 + \left|\dfrac{3}{z}\right|$

$\Rightarrow |z| - \dfrac{3}{|z|} \leq 2$

$\Rightarrow |z|^2 - 2|z| - 3 \leq 0 \Rightarrow (|z| + 1)(|z| - 3) \leq 0$
$\Rightarrow |z| > -1, |z| \leq 3.$
So, greatest value of $|z|$ is 3

103. (d): According to question,

$\tan 33° + \tan 12° = \dfrac{n}{m}$

$\tan 33° \times \tan 12° = \dfrac{k}{m}$

Now, $\dfrac{2m + n + k}{m} = 2 + \dfrac{n}{m} + \dfrac{k}{m}$

$\qquad = 2 + (\tan 33° + \tan 12°) + (\tan 33° \times \tan 12°)$
$\qquad = 2 + 1 = 3 \quad [\because \tan(33° + 12°) = \tan 45° = 1]$

104. (a): According to question, $\alpha\beta = \dfrac{-1}{4}$

$\Rightarrow \beta = -\dfrac{1}{4\alpha}$

105. (e): According to question,
$\alpha + \alpha^2 = 6$ ...(i) and $\alpha^3 = c$ ...(ii)
$\Rightarrow \alpha^2 + \alpha - 6 = 0$ (From (i))
$\Rightarrow (\alpha + 3)(\alpha - 2) = 0 \Rightarrow \alpha = -3$ or $\alpha = 2$
$\therefore$ From (ii), we get $c = -27$ or 8

106. (c): Let the other root be β.

Then, $6\beta = 1 \Rightarrow \beta = \dfrac{1}{6}$

Also, $6 + \beta = \dfrac{b}{a} \Rightarrow \dfrac{b}{a} = 6 + \dfrac{1}{6} = \dfrac{37}{6}$

107. (c): Let α be the root of the given quadratic equation. Then, $\alpha \times \alpha = 4 \Rightarrow \alpha = \pm 2$

Again $2\alpha = -\dfrac{(a+3)}{2}$ ...(i)

When $\alpha = 2$, (i) becomes $4 = -\dfrac{(a+3)}{2} \Rightarrow a = -11$

When $\alpha = -2$, (i) becomes $-4 = -\dfrac{(a+3)}{2} \Rightarrow a = 5$

108. (c) : Consider the 11 terms of the G.P. as,

$\dfrac{a}{r^5}, \dfrac{a}{r^4}, \dfrac{a}{r^3}, \dfrac{a}{r^2}, \dfrac{a}{r}, a, ar, ar^2, ar^3, ar^4, ar^5$

Now, $T_6 = a = 2$

$\therefore$ Product of 11 terms $= a^{11} = 2^{11} = 2048$

109. (b): Let the five consecutive terms of the G.P. be,

$\dfrac{a}{r^2}, \dfrac{a}{r}, a, ar, ar^2$

According to question, $a^5 = \dfrac{243}{32} \Rightarrow a = \dfrac{3}{2}$

So, middle term $= \dfrac{3}{2}$

110. (a) : Since $a_1, a_2, a_3, a_4 \in$ A.P.

$\therefore \quad a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = d$

Now, $\dfrac{1}{\sqrt{a_1} + \sqrt{a_2}} + \dfrac{1}{\sqrt{a_2} + \sqrt{a_3}} + \dfrac{1}{\sqrt{a_3} + \sqrt{a_4}}$

$= \dfrac{\sqrt{a_2} - \sqrt{a_1}}{a_2 - a_1} + \dfrac{\sqrt{a_3} - \sqrt{a_2}}{a_3 - a_2} + \dfrac{\sqrt{a_4} - \sqrt{a_3}}{a_4 - a_3}$

$= \dfrac{\sqrt{a_2} - \sqrt{a_1} + \sqrt{a_3} - \sqrt{a_2} + \sqrt{a_4} - \sqrt{a_3}}{d} = \dfrac{\sqrt{a_4} - \sqrt{a_1}}{a_3 - a_2}$

111. (d): $S_n = \dfrac{n}{2}(a + l) \quad \therefore \quad S_{20} = \dfrac{20}{2}(45) = 450$

$(\because a_1 + a_{20} = 45)$

112. (e) : We have, $1(1) + 2(1+3) + 3(1+3+5) + ...$
$\qquad\qquad\qquad + 10(1 + 3 + 5 + ... + 19)$

$= 1(1^2) + 2(2^2) + 3(3^2) + ... + 10(10^2)$

$= 1^3 + 2^3 + 3^3 + ... + 10^3$

$= \left[\dfrac{10(10+1)}{2}\right]^2 = (55)^2 = 3025$

113. (c) : We have, $a + 5d = 52$ ...(i)

$a + 10d = 112$ ...(ii)

Solving (i) and (ii), we get $d = 12$

114. (c) : Coefficient of $x^3 = {}^9C_3(3)^6 \cdot (k)^3$ ...(i)

Coefficient of $x^4 = {}^9C_4(3)^5 (k)^4$ ...(ii)

According to question, ${}^9C_3\, 3^6 \cdot k^3 = {}^9C_4\, 3^5 \cdot k^4$

$\Rightarrow \quad k = \dfrac{{}^9C_3 \cdot 3^6}{{}^9C_4 \cdot 3^5} = 2$

115. (a) : Total no. of ways $= 6 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
$\qquad\qquad\qquad\qquad = 4320$

116. (d): We have, ${}^nP_4 = 5\, {}^nP_3$

$\Rightarrow \quad \dfrac{n!}{(n-4)!} = 5\dfrac{n!}{(n-3)!} \Rightarrow n - 3 = 5 \Rightarrow n = 8$

117. (a) : We have, $2^{2016} = (2^6)^{336} = (64)^{336}$
$\qquad\qquad\qquad\qquad = (63 + 1)^{336}$

So, remainder $= 1$

118. (c) : We have, ${}^nC_2 + {}^nC_3 = {}^6C_3$

$\Rightarrow \quad \dfrac{n!}{(n-2)!\, 2!} + \dfrac{n!}{(n-3)!\, 3!} = \dfrac{6!}{3!\, 3!}$

$\Rightarrow \quad \dfrac{n(n-1)}{2} + \dfrac{n(n-1)(n-2)}{6} = 20$

$\Rightarrow \quad n(n-1)(n+1) = 120$

$\Rightarrow \quad (n+1)n(n-1) = 6 \times 5 \times 4$

$\therefore \quad n = 5$

Now, ${}^5C_x = {}^5C_3 \Rightarrow x = 3$ or $x = 2 \quad (\because {}^nC_r = {}^nC_{n-r})$

119. (b): We have,

$\displaystyle\sum_{k=0}^{18} \dfrac{k}{{}^{18}C_k} = a \sum_{k=0}^{18} \dfrac{1}{{}^{18}C_k}$

$\Rightarrow \quad 0 + \dfrac{1}{{}^{18}C_1} + \dfrac{2}{{}^{18}C_2} + ... + \dfrac{18}{{}^{18}C_{18}}$

$\qquad\qquad = a\left(\dfrac{1}{{}^{18}C_0} + \dfrac{1}{{}^{18}C_1} + ... + \dfrac{1}{{}^{18}C_{18}}\right)$

$\Rightarrow \quad 18\left(\dfrac{1}{{}^{18}C_0} + \dfrac{1}{{}^{18}C_1} + ... + \dfrac{1}{{}^{18}C_8}\right) + \dfrac{9}{{}^{18}C_9}$

$\qquad = 2a\left(\dfrac{1}{{}^{18}C_0} + \dfrac{1}{{}^{18}C_1} + ... + \dfrac{1}{{}^{18}C_8}\right) + \dfrac{a}{{}^{18}C_9}$

On comparing, we get $2a = 18 \Rightarrow a = 9$

120. (b): According to question,

$\Rightarrow \begin{bmatrix} a & b \\ a & -a \end{bmatrix}\begin{bmatrix} a & b \\ a & -a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a^2 + ab & 0 \\ 0 & ab + a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

On comparing, we get $a^2 + ab = 1$

$\Rightarrow \quad b = \dfrac{1 - a^2}{a}$

■ ■

SOLVED PAPER 2016
KARNATAKA CET

1. The set A has 4 elements and the set B has 5 elements then the number of injective mappings that can be defined from A to B is
   (a) 144
   (b) 72
   (c) 60
   (d) 120

2. Let $f : R \to R$ be defined by $f(x) = 2x + 6$ which is a bijective mapping then $f^{-1}(x)$ is given by
   (a) $\dfrac{x}{2} - 3$
   (b) $2x + 6$
   (c) $x - 3$
   (d) $6x + 2$

3. Let $*$ be a binary operation defined on R by $a * b = \dfrac{a + b}{4} \ \forall \ a, b \in R$ then the operation $*$ is
   (a) Commutative and Associative
   (b) Commutative but not Associative
   (c) Associative but not Commutative
   (d) Neither Associative nor Commutative

4. The value of $\sin^{-1}\left(\cos\dfrac{53\pi}{5}\right)$ is
   (a) $\dfrac{3\pi}{5}$
   (b) $\dfrac{-3\pi}{5}$
   (c) $\dfrac{\pi}{10}$
   (d) $\dfrac{-\pi}{10}$

5. If $3 \tan^{-1}x + \cot^{-1}x = \pi$ then x equal to
   (a) 0
   (b) 1
   (c) –1
   (d) $\dfrac{1}{2}$

6. The simplified form of $\tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}\left(\dfrac{x-y}{x+y}\right)$ is equal to
   (a) 0
   (b) $\dfrac{\pi}{4}$
   (c) $\dfrac{\pi}{2}$
   (d) π

7. If x, y, z are all different and not equal to zero and $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$ then the value of $x^{-1} + y^{-1} + z^{-1}$ is equal to
   (a) xyz
   (b) $x^{-1}y^{-1}z^{-1}$
   (c) $-x - y - z$
   (d) -1

8. If A is any square matrix of order 3×3 then $|3A|$ is equal to
   (a) $3|A|$
   (b) $\dfrac{1}{3}|A|$
   (c) $27\,|A|$
   (d) $9\,|A|$

9. If $y = e^{\sin^{-1}(t^2-1)}$ & $x = e^{\sec^{-1}\left(\frac{1}{t^2-1}\right)}$ then $\dfrac{dy}{dx}$ is equal to
   (a) $\dfrac{x}{y}$
   (b) $\dfrac{-y}{x}$
   (c) $\dfrac{y}{x}$
   (d) $\dfrac{-x}{y}$

10. If $A = \dfrac{1}{\pi}\begin{bmatrix} \sin^{-1}(\pi x) & \tan^{-1}\left(\dfrac{x}{\pi}\right) \\ \sin^{-1}\left(\dfrac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$,
    $B = \dfrac{1}{\pi}\begin{bmatrix} -\cos^{-1}(\pi x) & \tan^{-1}\left(\dfrac{x}{\pi}\right) \\ \sin^{-1}\left(\dfrac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$
    then $A - B$ is equal to
    (a) I
    (b) 0
    (c) $2I$
    (d) $\dfrac{1}{2}I$

11. If $x^y = e^{x-y}$ then $\dfrac{dy}{dx}$ is equal to
    (a) $\dfrac{\log x}{\log(x-y)}$
    (b) $\dfrac{e^x}{x^{x-y}}$
    (c) $\dfrac{\log x}{(1+\log x)^2}$
    (d) $\dfrac{1}{y} - \dfrac{1}{x-y}$

12. If A is a matrix of order $m \times n$ and B is a matrix such that AB' and $B'A$ are both defined, the order of the matrix B is
    (a) $m \times m$
    (b) $n \times n$
    (c) $n \times m$
    (d) $m \times n$

13. The value of $\int \dfrac{e^x(1+x)\,dx}{\cos^2(e^x \cdot x)}$ is equal to
    (a) $-\cot(e\,x^x) + c$
    (b) $\tan(e^x \cdot x) + c$
    (c) $\tan(e^x) + c$
    (d) $\cot(e^x) + c$

14. If xyz are not equal and $\neq 0$, $\neq 1$ the value of $\begin{vmatrix} \log x & \log y & \log z \\ \log 2x & \log 2y & \log 2z \\ \log 3x & \log 3y & \log 3z \end{vmatrix}$ is equal to
    (a) $\log(xyz)$
    (b) $\log(6xyz)$
    (c) 0
    (d) $\log(x + y + z)$

15. The function $f(x) = [x]$ where $[x]$ is the greatest integer function is continuous at
    (a) 1.5   (b) 4   (c) 1   (d) -2

16. The value of $\int \dfrac{e^x(x^2 \tan^{-1} x + \tan^{-1} x + 1)}{x^2 + 1} dx$ is equal to
    (a) $e^x \tan^{-1} x + c$
    (b) $\tan^{-1}(e^x) + c$
    (c) $\tan^{-1}(x^e) + c$
    (d) $e^{\tan^{-1} x} + c$

17. If $2\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}|$ then the angle between $\vec{a}$ & $\vec{b}$ is
    (a) 30°   (b) 0°   (c) 90°   (d) 60°

18. If $x^m y^n = (x + y)^{m+n}$ then $\dfrac{dy}{dx}$ is equal to
    (a) $\dfrac{x + y}{xy}$ (b) xy (c) 0   (d) $\dfrac{y}{x}$

19. The general solution of $\cot\theta + \tan\theta = 2$ is
    (a) $\theta = \dfrac{n\pi}{2} + (-1)^n \dfrac{\pi}{8}$
    (b) $\dfrac{n\pi}{2} + (-1)^n \dfrac{\pi}{4}$
    (c) $\theta = \dfrac{n\pi}{2} + (-1)^n \dfrac{\pi}{6}$
    (d) $\theta = n\pi + (-1)^n \dfrac{\pi}{8}$

20. The value of $\int\limits_{-\pi/4}^{\pi/4} \sin^{103} x \cdot \cos^{101} x \, dx$ is
    (a) $\left(\dfrac{\pi}{4}\right)^{103}$
    (b) $\left(\dfrac{\pi}{4}\right)^{101}$
    (c) 2
    (d) 0

21. The length of latus rectum of the parabola $4y^2 + 3x + 3y + 1 = 0$ is
    (a) $\dfrac{4}{3}$ (b) 7   (c) 12   (d) $\dfrac{3}{4}$

22. The value of $\int \dfrac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}} dx$ is equal to
    (a) 0   (b) $\dfrac{x^3}{3}$ (c) $\dfrac{3}{x^3}$ (d) $\dfrac{1}{x}$

23. The differential coefficient of $\log_{10} x$ with respect to $\log_x 10$ is
    (a) 1
    (b) $-(\log_{10} x)^2$
    (c) $(\log_x 10)^2$
    (d) $\dfrac{x^2}{100}$

24. The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is
    (a) $\dfrac{22}{7}$ (b) $\dfrac{6}{7}$ (c) $\dfrac{7}{6}$ (d) $\dfrac{-6}{7}$

25. The real part of $(1 - \cos\theta + i \sin\theta)^{-1}$ is
    (a) $\dfrac{1}{2}$
    (b) $\dfrac{1}{1 + \cos\theta}$
    (c) $\tan\dfrac{\theta}{2}$
    (d) $\cot\dfrac{\theta}{2}$

26. $\int\limits_{0}^{\pi/2} \dfrac{\sin^{1000} x \, dx}{\sin^{1000} x + \cos^{1000} x}$ is equal to
    (a) 1000
    (b) 1
    (c) $\dfrac{\pi}{2}$
    (d) $\dfrac{\pi}{4}$

27. If $1 + \sin\theta + \sin^2\theta + \dots$ upto $\infty = 2\sqrt{3} + 4$, then $\theta = $ _____
    (a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{3}$ (d) $\dfrac{3\pi}{4}$

28. $\lim\limits_{x \to 0} \dfrac{xe^x - \sin x}{x}$ is equal to
    (a) 3   (b) 1   (c) 0   (d) 2

29. If $\tan^{-1}(x^2 + y^2) = \alpha$ then $\dfrac{dy}{dx}$ is equal to
    (a) $\dfrac{-x}{y}$ (b) xy (c) $\dfrac{x}{y}$ (d) $-xy$

30. The simplified form of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is
    (a) 0   (b) 1   (c) -1 (d) i

31. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2 y - y^3 = 2$
    (a) Touch each other
    (b) Cut each other at right angle
    (c) Cut at an angle $\dfrac{\pi}{3}$
    (d) Cut at an angle $\dfrac{\pi}{4}$

32. The equation of the normal to the curve $y(1 + x^2) = 2 - x$ where the tangent crosses x-axis is
    (a) $5x - y - 10 = 0$
    (b) $x - 5y - 10 = 0$
    (c) $5x + y + 10 = 0$
    (d) $x + 5y + 10 = 0$

33. The maximum value of $\left(\dfrac{1}{x}\right)^x$ is
    (a) e (b) e^e (c) $e^{\frac{1}{e}}$ (d) $\left(\dfrac{1}{e}\right)^e$

34. The solution for the differential equation

$$\frac{dy}{y} + \frac{dx}{x} = 0 \text{ is}$$

(a) $\dfrac{1}{y} + \dfrac{1}{x} = c$ (b) $\log x \cdot \log y = c$

(c) $xy = c$ (d) $x + y = c$

35. The order and degree of the differential equation

$$\left[1 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right)\right]^{3/4} = \frac{d^2y}{dx^2}$$

(a) order = 2, degree = 3   (b) order = 2, degree = 4

(c) order = 2, degree = $\dfrac{3}{4}$ (d) order = 2, degree = not defined

36. If $\vec{a}$ and $\vec{b}$ are unit vectors then what is the angle between $\vec{a}$ and $\vec{b}$ for $\sqrt{3}\,\vec{a} - \vec{b}$ to be unit vector?

(a) 30°   (b) 45°   (c) 60°   (d) 90°

37. The sum of 1$^{\text{st}}$ n terms of the series

$$\frac{1^2}{1} + \frac{1^2 + 2^2}{1+2} + \frac{1^2 + 2^2 + 3^2}{1+2+3} + \dots$$

(a) $\dfrac{n+2}{3}$ (b) $\dfrac{n(n+2)}{3}$

(c) $\dfrac{n(n-2)}{3}$ (d) $\dfrac{n(n-2)}{6}$

38. The 11$^{\text{th}}$ term in the expansion of $\left(x + \dfrac{1}{\sqrt{x}}\right)^{14}$ is

(a) $\dfrac{999}{x}$ (b) $\dfrac{1001}{x}$

(c) i (d) $\dfrac{x}{1001}$

39. Suppose $\vec{a} + \vec{b} + \vec{c} = 0, |\vec{a}| = 3, |\vec{b}| = 5, |\vec{c}| = 7$, then the angle between $\vec{a}$ & $\vec{b}$ is

(a) π (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{4}$

40. If $a = 3, b = 4, c = 5$ each one of $\vec{a}, \vec{b}$ & $\vec{c}$ is perpendicular to the sum of the remaining then $|\vec{a} + \vec{b} + \vec{c}|$ is equal to

(a) $\dfrac{5}{\sqrt{2}}$ (b) $\dfrac{2}{\sqrt{5}}$ (c) $5\sqrt{2}$ (d) $\sqrt{5}$

41. If the straight lines $2x + 3y - 3 = 0$ and $x + ky + 7 = 0$ are perpendicular, then the value of k is

(a) $\dfrac{2}{3}$ (b) $\dfrac{3}{2}$ (c) $-\dfrac{2}{3}$ (d) $-\dfrac{3}{2}$

42. The rate of change of area of a circle with respect to its radius at $r = 2$ cms is

(a) 4   (b) 2π (c) 2   (d) 4π

43. The value of $\tan\dfrac{\pi}{8}$ is equal to

(a) $\dfrac{1}{2}$ (b) $\sqrt{2} + 1$

(c) $\dfrac{1}{\sqrt{2}+1}$ (d) $1 - \sqrt{2}$

44. Area lying between the curves $y^2 = 2x$ and $y = x$ is

(a) $\dfrac{2}{3}$ sq. units   (b) $\dfrac{1}{3}$ sq. units

(c) $\dfrac{1}{4}$ sq. units   (d) $\dfrac{3}{4}$ sq. units

45. If $P(A \cap B) = \dfrac{7}{10}$ and $P(B) = \dfrac{17}{20}$, where P stands for probability then $P(A/B)$ is equal to

(a) $\dfrac{7}{8}$ (b) $\dfrac{17}{20}$ (c) $\dfrac{14}{17}$ (d) $\dfrac{1}{8}$

46. The coefficient of variation of two distributions are 60 and 70. The standard deviation are 21 and 16 respectively, then their mean is

(a) 35   (b) 23   (c) 28.25   (d) 22.85

47. Two cards are drawn at random from a pack of 52 cards. The probability of these two being "Aces" is

(a) $\dfrac{1}{26}$ (b) $\dfrac{1}{221}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{1}{13}$

48. If $\sin^{-1} x + \sin^{-1} y = \dfrac{\pi}{2}$, then x^2 is equal to

(a) $1 - y^2$ (b) y^2 (c) 0   (d) $\sqrt{1-y}$

49. The value of $\displaystyle\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}\, dx$ is

(a) 10   (b) 0   (c) 8   (d) 3

50. The contrapositive of the converse of the statement "If x is a prime number then x is odd" is

(a) If x is not a prime number then x is odd.

(b) If x is not an odd number then x is not a prime number.

(c) If x is a prime number then it is not odd.

(d) If x is not a prime number then x is not an odd.

51. Two dice are thrown simultaneously, the probability of obtaining a total score of 5 is

(a) $\dfrac{1}{18}$ (b) $\dfrac{1}{12}$ (c) $\dfrac{1}{9}$ (d) $\dfrac{1}{6}$

52. If $A = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$ and $A + A^T = I$, where I is the unit matrix of 2×2 & A^T is the transpose of A, then the value of θ is equal to

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{3}$ (c) π (d) $\dfrac{3\pi}{2}$

53. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ then $A^2 - 5A$ is equal to

(a) I (b) $-I$ (c) $7I$ (d) $-7I$

54. The value of x if $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector is

(a) $\pm\dfrac{1}{\sqrt{3}}$ (b) $\pm\sqrt{3}$

(c) ± 3 (d) $\pm\dfrac{1}{3}$

55. If $x = 2 + 3\cos\theta$ and $y = 1 - 3\sin\theta$ represent a circle then the centre and radius is

(a) $(2, 1), 9$ (b) $(2, 1), 3$

(c) $(1, 2), \dfrac{1}{3}$ (d) $(-2, -1), 3$

56. The vector equation of the plane which is at a distance $\dfrac{3}{\sqrt{14}}$ from the origin and the normal from the origin is $2\hat{i} - 3\hat{j} + \hat{k}$ is

(a) $\vec{r} \cdot (2\hat{i} - 3\hat{j} + \hat{k}) = 3$
(b) $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 9$
(c) $\vec{r} \cdot (\hat{i} + 2\hat{j}) = 3$
(d) $\vec{r} \cdot (2\hat{i} + \hat{k}) = 3$

57. Find the co-ordinates of the foot of the perpendicular drawn from the origin to the plane $5y + 8 = 0$.

(a) $\left(0, -\dfrac{18}{5}, 2\right)$ (b) $\left(0, \dfrac{8}{5}, 2\right)$

(c) $\left(\dfrac{8}{25}, 0, 0\right)$ (d) $\left(0, -\dfrac{8}{5}, 0\right)$

58. If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of a vector $\vec{a}$, then $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$ is equal to

(a) 2 (b) 3 (c) -1 (d) 0

59. The value of the $\sin 1° + \sin 2° + \dots + \sin 359°$ is equal to

(a) 0 (b) 1 (c) -1 (d) 180

60. Integrating factor of $x\dfrac{dy}{dx} - y = x^4 - 3x$ is

(a) x (b) $\log x$ (c) $\dfrac{1}{x}$ (d) $-x$

SOLUTIONS

1. (d): Since, set A has 4 elements and set B has 5 elements, then the number of injective mappings from A to $B = {}^5P_4 = 120$

2. (a): We have, $f(x) = 2x + 6$
Since, f is a bijective function
$\therefore$ f^{-1} exists.
Let $x \in R$ then there exist $y \in R$ such that $f(x) = y$
$\Rightarrow 2x + 6 = y \Rightarrow x = \dfrac{y - 6}{2} \Rightarrow f^{-1}(y) = \dfrac{y - 6}{2}$
$\therefore f^{-1}(x) = \dfrac{x - 6}{2}$ for all $x \in R$

3. (b): Commutativity : Let $a, b \in R$, then
$a * b = \dfrac{a + b}{4} = \dfrac{b + a}{4} = b * a$
$\therefore$ $*$ is commutative.

Associativity : Let $a, b, c \in R$, then
$(a * b) * c = \left(\dfrac{a + b}{4}\right) * c = \dfrac{\left(\dfrac{a + b}{4}\right) + c}{4} = \dfrac{a + b + 4c}{16}$
and $a * (b * c) = a * \left(\dfrac{b + c}{4}\right)$
$= \dfrac{a + \left(\dfrac{b + c}{4}\right)}{4} = \dfrac{4a + b + c}{16}$
$\because (a * b) * c \neq a * (b * c)$
$\therefore$ $*$ is not associative.

4. (d): $\sin^{-1}\left(\cos\dfrac{53\pi}{5}\right) = \sin^{-1}\left\{\cos\left(10\pi + \dfrac{3\pi}{5}\right)\right\}$
$= \sin^{-1}\left\{\cos\left(\dfrac{3\pi}{5}\right)\right\} = \sin^{-1}\left\{\sin\left(\dfrac{\pi}{2} - \dfrac{3\pi}{5}\right)\right\}$
$= \sin^{-1}\left\{\sin\left(-\dfrac{\pi}{10}\right)\right\} = -\dfrac{\pi}{10}$

5. (b): $3\tan^{-1}x + \cot^{-1}x = \pi$ (Given)
$\Rightarrow \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right) + \tan^{-1}\left(\dfrac{1}{x}\right) = \pi$
$\Rightarrow \tan^{-1}\left\{\dfrac{\dfrac{3x - x^3}{1 - 3x^2} + \dfrac{1}{x}}{1 - \left(\dfrac{3x - x^3}{1 - 3x^2}\right)\left(\dfrac{1}{x}\right)}\right\} = \pi$

$\Rightarrow \dfrac{1 - x^4}{-2x - 2x^3} = \tan \pi = 0$

$\Rightarrow 1 - x^4 = 0 \Rightarrow x^4 = 1$

$\Rightarrow x = 1$ ($\because x = -1$ doesn't satisfy the given equation)

6. (b): $\tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}\left(\dfrac{x - y}{x + y}\right)$

$= \tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}\left\{\dfrac{\dfrac{x}{y} - 1}{\dfrac{x}{y} + 1}\right\}$

$= \tan^{-1}\left(\dfrac{x}{y}\right) - \left[\tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}(1)\right]$

$= \tan^{-1}(1) = \dfrac{\pi}{4}$

7. (d): We have, $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$

Applying $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$, we get

$\begin{vmatrix} x & 0 & -z \\ 0 & y & -z \\ 1 & 1 & 1+z \end{vmatrix} = 0$

Expanding along R_1, we get

$x(y + yz + z) + zy = 0 \Rightarrow xy + xyz + zx + zy = 0$

Dividing both sides by xyz, we get

$\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = -1$ \qquad ($\because$ $x, y, z \neq 0$)

8. (c): Since, A is any square matrix of order 3.

$\Rightarrow |3A| = 3^3|A| = 27 |A|$

[$\because$ For a square matrix A of order n, we have $|kA| = k^n|A|$]

9. (b): We have, $y = e^{\sin^{-1}(t^2 - 1)}$

Differentiating w.r.t. t, we get

$\dfrac{dy}{dt} = e^{\sin^{-1}(t^2 - 1)} \times \dfrac{2t}{\sqrt{1 - (t^2 - 1)^2}}$ \qquad ...(i)

Also, $x = e^{\sec^{-1}\left(\frac{1}{t^2 - 1}\right)} = e^{\cos^{-1}(t^2 - 1)}$

Differentiating w.r.t. t, we get

$\dfrac{dx}{dt} = e^{\cos^{-1}(t^2 - 1)} \times \dfrac{-(2t)}{\sqrt{1 - (t^2 - 1)^2}}$ \qquad ...(ii)

$\therefore$ From (i) & (ii), we get

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{e^{\sin^{-1}(t^2 - 1)}}{e^{\sec^{-1}\left(\frac{1}{t^2 - 1}\right)}} \times \dfrac{\dfrac{2t}{\sqrt{1 - (t^2 - 1)^2}}}{\left(\dfrac{-2t}{\sqrt{1 - (t^2 - 1)^2}}\right)} = \dfrac{-y}{x}$

10. (d): $A - B = \dfrac{1}{\pi}\begin{bmatrix} \sin^{-1}(\pi x) & \tan^{-1}\left(\dfrac{x}{\pi}\right) \\ \sin^{-1}\left(\dfrac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$

$\qquad - \dfrac{1}{\pi}\begin{bmatrix} -\cos^{-1}(\pi x) & \tan^{-1}\left(\dfrac{x}{\pi}\right) \\ \sin^{-1}\left(\dfrac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$

$= \dfrac{1}{\pi}\begin{bmatrix} \sin^{-1}(\pi x) + \cos^{-1}(\pi x) & 0 \\ 0 & \cot^{-1}(\pi x) + \tan^{-1}(\pi x) \end{bmatrix}$

$= \dfrac{1}{\pi}\begin{bmatrix} \dfrac{\pi}{2} & 0 \\ 0 & \dfrac{\pi}{2} \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \dfrac{1}{2}I$

11. (c): We have, $x^y = e^{x - y}$

Taking log on both sides, we get

$y \log x = x - y$ \qquad ...(i)

Differentiating (i) w.r.t. x, we get

$y \cdot \dfrac{1}{x} + \log x\left(\dfrac{dy}{dx}\right) = 1 - \dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx}(\log x + 1) = 1 - \dfrac{y}{x}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x - y}{x(1 + \log x)} = \dfrac{y \log x}{(y \log x + y)(1 + \log x)}$

\hfill [From (i)]

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\log x}{(1 + \log x)^2}$

12. (d)

13. (b): Let $I = \displaystyle\int \dfrac{e^x(1 + x)}{\cos^2(e^x \cdot x)}dx$

Put $xe^x = t \Rightarrow e^x(x + 1)dx = dt$

$\therefore I = \displaystyle\int \dfrac{dt}{\cos^2 t} = \int \sec^2 t\, dt = \tan t + c$

$= \tan(x \cdot e^x) + c$

14. (c) : Let $\Delta = \begin{vmatrix} \log x & \log y & \log z \\ \log 2x & \log 2y & \log 2z \\ \log 3x & \log 3y & \log 3z \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} \log x & \log y & \log z \\ \log 2 & \log 2 & \log 2 \\ \log 3 & \log 3 & \log 3 \end{vmatrix}$$

Taking out $\log 2$ and $\log 3$ from R_2 and R_3 respectively, we get

$$\Delta = (\log 2)(\log 3) \begin{vmatrix} \log x & \log y & \log z \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

15. (a)

16. (a) : Let $I = \int e^x \left(\dfrac{x^2 \tan^{-1} x + \tan^{-1} x + 1}{x^2 + 1} \right) dx$

$= \int e^x \left(\dfrac{((1+x^2)\tan^{-1} x + 1)}{1+x^2} \right) dx$

$= \int e^x \left(\tan^{-1} x + \dfrac{1}{1+x^2} \right) dx$

$= \int e^x \tan^{-1} x\, dx + \int e^x \cdot \dfrac{1}{1+x^2} dx$

$= \tan^{-1} x \cdot e^x - \int \dfrac{1}{1+x^2} \cdot e^x dx + \int e^x \cdot \dfrac{1}{1+x^2} dx + c$

$= e^x \tan^{-1} x + c$

17. (d) : We have, $2\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}|$
$\Rightarrow 2|\vec{a}| \cdot |\vec{b}| \cos\theta = |\vec{a}| \cdot |\vec{b}|$
$\Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = 60°$

18. (d) : We have, $x^m y^n = (x+y)^{m+n}$ (Given)
Taking log on both sides, we get
$m \log x + n \log y = (m+n) \log(x+y)$
Differentiating w.r.t. x, we get
$m \cdot \dfrac{1}{x} + n \cdot \dfrac{1}{y} \dfrac{dy}{dx} = (m+n) \cdot \left(\dfrac{1}{x+y} \right) \left(1 + \dfrac{dy}{dx} \right)$

$\Rightarrow \left(\dfrac{m}{x} - \dfrac{m+n}{x+y} \right) = \dfrac{dy}{dx} \left(\dfrac{m+n}{x+y} - \dfrac{n}{y} \right)$

$\Rightarrow \dfrac{my - nx}{x(x+y)} = \dfrac{dy}{dx} \left(\dfrac{my - nx}{y(x+y)} \right)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x}$

19. (b) : We have, $\cot\theta + \tan\theta = 2$ (Given)

$\Rightarrow \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta} = 2 \Rightarrow \dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = 2$

$\Rightarrow 2\sin\theta\cos\theta = 1 \Rightarrow \sin 2\theta = \sin\pi/2$

$\Rightarrow 2\theta = n\pi + (-1)^n \dfrac{\pi}{2} \Rightarrow \theta = \dfrac{n\pi}{2} + (-1)^n \dfrac{\pi}{4}$

20. (d) : We have, $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^{103} x \cdot \cos^{101} x\, dx$

Since, integral is product of an even and an odd function so the function is odd.
$\therefore$ Given integral will be zero.

21. (d) : Given equation of parabola is
$4y^2 + 3x + 3y + 1 = 0$

$\Rightarrow y^2 + \dfrac{3}{4} x + \dfrac{3}{4} y + \dfrac{1}{4} = 0$

$\Rightarrow \left(y + \dfrac{3}{8} \right)^2 - \dfrac{9}{64} + \dfrac{3}{4} x + \dfrac{1}{4} = 0$

$\Rightarrow \left(y + \dfrac{3}{8} \right)^2 = -\dfrac{3}{4} \left(x + \dfrac{7}{48} \right)$

$\Rightarrow$ Length of latus rectum $= \dfrac{3}{4}$

22. (b) : Let $I = \int \dfrac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}} dx$

$= \int \dfrac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}} dx = \int \left(\dfrac{x^6 - x^5}{x^4 - x^3} \right) dx$

$= \int \dfrac{x^5}{x^3} \left[\dfrac{x-1}{x-1} \right] dx = \int x^2 dx = \dfrac{x^3}{3} + C$

23. (b) : Let $\log_x 10 = t \Rightarrow t = \dfrac{\log 10}{\log x}$ ...(i)

$\dfrac{d}{d \log_x 10} [\log_{10} x] = \dfrac{d}{dt} \left[\dfrac{\log x}{\log 10} \right] = \dfrac{d}{dt} \left(\dfrac{1}{t} \right)$ [From (i)]

$= -\dfrac{1}{t^2} = -[\log_x 10]^{-2} = -[\log_{10} x]^2$

24. (b) : We have, $x = t^2 + 3t - 8, \; y = 2t^2 - 2t - 5$

Slope of tangent to the curve $= \dfrac{dy}{dx}$

Now, $\dfrac{dy}{dt} = 4t - 2, \; \dfrac{dx}{dt} = 2t + 3$

$\therefore \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t - 2}{2t + 3}$

When $x = 2$, $y = -1$, we get $t = 2$

$\therefore \left(\dfrac{dy}{dx}\right)_{t=2} = \dfrac{8-2}{4+3} = \dfrac{6}{7}$

25. (a): $(1 - \cos\theta + i\sin\theta)^{-1} = \dfrac{1}{1 - \cos\theta + i\sin\theta}$

$= \dfrac{1}{1 - \cos\theta + i\sin\theta} \times \dfrac{1 - \cos\theta - i\sin\theta}{1 - \cos\theta - i\sin\theta}$

$= \dfrac{1 - \cos\theta - i\sin\theta}{1 + \cos^2\theta - 2\cos\theta + \sin^2\theta}$

$= \dfrac{(1 - \cos\theta) - i\sin\theta}{2(1 - \cos\theta)}$

$\therefore$ Real part of given expression is $\dfrac{1}{2}$

26. (d): Let $I = \displaystyle\int_0^{\pi/2} \dfrac{\sin^{1000} x}{\sin^{1000} x + \cos^{1000} x}\, dx$ ...(i)

$= \displaystyle\int_0^{\pi/2} \dfrac{\sin^{1000}\left(\dfrac{\pi}{2} - x\right)}{\sin^{1000}\left(\dfrac{\pi}{2} - x\right) + \cos^{1000}\left(\dfrac{\pi}{2} - x\right)}\, dx$

$\Rightarrow I = \displaystyle\int_0^{\pi/2} \dfrac{\cos^{1000} x}{\sin^{1000} x + \cos^{1000} x}\, dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \displaystyle\int_0^{\pi/2} dx = (x)\Big|_0^{\pi/2} = \dfrac{\pi}{2}$

$\therefore\ I = \dfrac{\pi}{4}$

27. (c): $1 + \sin\theta + \sin^2\theta + \ldots$ upto $\infty = 2\sqrt{3} + 4$

Since, R.H.S is an infinite geometric progression with common ratio $\sin\theta$.

$\therefore\ \dfrac{1}{1 - \sin\theta} = 2\sqrt{3} + 4$

$\Rightarrow 2\sqrt{3} + 4 - 2\sqrt{3}\sin\theta - 4\sin\theta = 1$

$\Rightarrow 2\sqrt{3} + 3 = \sin\theta(2\sqrt{3} + 4)$

$\Rightarrow \sin\theta = \dfrac{2\sqrt{3} + 3}{2\sqrt{3} + 4} \times \dfrac{2\sqrt{3} - 4}{2\sqrt{3} - 4} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \theta = \dfrac{\pi}{3}$

28. (c): $\displaystyle\lim_{x\to 0} \dfrac{xe^x - \sin x}{x} = \lim_{x\to 0}\left[\dfrac{xe^x}{x} - \dfrac{\sin x}{x}\right]$

$= \displaystyle\lim_{x\to 0} e^x - \lim_{x\to 0}\left(\dfrac{\sin x}{x}\right) = 1 - 1 = 0$

29. (a): We have, $\tan^{-1}(x^2 + y^2) = \alpha$

$\Rightarrow x^2 + y^2 = \tan\alpha$

Differentiating w.r.t. x, we get

$2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{-x}{y}$

30. (a): $i^n + i^{n+1} + i^{n+2} + i^{n+3}$

$= i^n + i^{n+1} + i^n \cdot i^2 + i^{n+1} \cdot i^2$

$= i^n + i^{n+1} - i^n - i^{n+1}$ $[\because\ i^2 = -1]$

$= 0$

31. (b): Given curves are $x^3 - 3xy^2 + 2 = 0$ ...(i)

and $3x^2y - y^3 = 2$ ...(ii)

Differentiating (i) w.r.t. x, we get

$3x^2 - 3\left[x(2y)\dfrac{dy}{dx} + y^2\right] = 0$

$\Rightarrow x^2 - 2xy\dfrac{dy}{dx} - y^2 = 0$

$\Rightarrow m_1 = \dfrac{dy}{dx} = \dfrac{x^2 - y^2}{2xy}$ ...(iii)

Differentiating (ii) w.r.t. x, we get

$3\left(x^2\dfrac{dy}{dx} + y(2x)\right) - 3y^2\dfrac{dy}{dx} = 0$

$\Rightarrow (x^2 - y^2)\dfrac{dy}{dx} = -2xy$

$\Rightarrow m_2 = \dfrac{dy}{dx} = -\dfrac{2xy}{x^2 - y^2}$ ...(iv)

$\because\ m_1 m_2 = -1$ [From (iii) & (iv)]

Hence, given curves cut each other at right angle.

32. (a): The equation of given curve is

$y(1 + x^2) = 2 - x$ ...(i)

Since, the curve crosses x-axis

$\therefore\ y = 0$ at that point

$\Rightarrow 0 = 2 - x \Rightarrow x = 2$

So, point of contact is $(2, 0)$

Differentiate (i) w.r.t. x, we get

$y(2x) + (1 + x^2)\dfrac{dy}{dx} = -1 \Rightarrow \dfrac{dy}{dx} = \dfrac{-1 - 2xy}{1 + x^2}$

$\therefore \left(\dfrac{dy}{dx}\right)_{\text{at}(2,0)} = \dfrac{-1}{1+4} = \dfrac{-1}{5}$

$\therefore$ Equation of normal to the given curve is

$y - 0 = 5(x - 2) \Rightarrow 5x - y - 10 = 0$

33. (c): Let $y = \left(\dfrac{1}{x}\right)^x$

Taking log on both sides, we get

$\log y = x \log\left(\dfrac{1}{x}\right) = -x \log x$

Differentiate w.r.t x, we get

$\dfrac{1}{y}\dfrac{dy}{dx} = -x \cdot \dfrac{1}{x} + \log x(-1) = -1 - \log x$

$\Rightarrow \dfrac{dy}{dx} = -y(1 + \log x)$ ....(i)

For maximum / minimum, $\dfrac{dy}{dx} = 0 \Rightarrow x = \dfrac{1}{e}$

Differentiate (i) w.r.t. x, we get

$\dfrac{d^2 y}{dx^2} = \dfrac{-y}{x} - (1 + \log x)\dfrac{dy}{dx}$

$\Rightarrow \dfrac{d^2 y}{dx^2} < 0$ at $x = \dfrac{1}{e}$

$\therefore$ Maximum value of $\left(\dfrac{1}{x}\right)^x$ is $(e)^{1/e}$

34. (c) : We have, $\dfrac{dy}{y} + \dfrac{dx}{x} = 0$

$\Rightarrow \dfrac{dy}{y} = -\dfrac{dx}{x}$

Integrating both sides, we get

$\Rightarrow \int \dfrac{dy}{y} = -\int \dfrac{dx}{x} \Rightarrow \log y = -\log x + \log c$

$\Rightarrow \log(yx) = \log c \Rightarrow xy = c$

35. (d): $\left[1 + \left(\dfrac{dy}{dx}\right)^2 + \sin\left(\dfrac{dy}{dx}\right)\right]^{3/4} = \dfrac{d^2 y}{dx^2}$

$\Rightarrow 1 + \left(\dfrac{dy}{dx}\right)^2 + \sin\left(\dfrac{dy}{dx}\right) = \left(\dfrac{d^2 y}{dx^2}\right)^{4/3}$

$\Rightarrow$ order = 2 and degree is not defined.

36. (a) : We have, $|\sqrt{3}\,\vec{a} - \vec{b}| = 1$

$\Rightarrow |\sqrt{3}\,\vec{a} - \vec{b}|^2 = 1$

$\Rightarrow (\sqrt{3}|\vec{a}|)^2 + |\vec{b}|^2 - 2\sqrt{3}(\vec{a}\cdot\vec{b}) = 1$

$\Rightarrow 3 + 1 - 2\sqrt{3}|\vec{a}||\vec{b}|\cos\theta = 1$

$\Rightarrow 3 - 2\sqrt{3}\cos\theta = 0$ $[\because |\vec{a}| = 1, |\vec{b}| = 1]$

$\Rightarrow \cos\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = 30°$

37. (b): Let $S_n = \dfrac{1^2}{1} + \dfrac{1^2 + 2^2}{1+2} + \dfrac{1^2 + 2^2 + 3^2}{1+2+3} + \ldots\ldots$

$T_n = \dfrac{1^2 + 2^2 + 3^2 + \ldots.. + n^2}{1 + 2 + 3 + \ldots.. + n}$

$= \dfrac{\Sigma n^2}{\Sigma n} = \dfrac{n(n+1)(2n+1)}{6} \times \dfrac{2}{n(n+1)} = \dfrac{2n+1}{3}$

$\therefore \; S_n = \dfrac{2\Sigma n + n}{3} = \dfrac{\dfrac{2n(n+1)}{2} + n}{3}$

$= \dfrac{n(n+1) + n}{3} = \dfrac{n^2 + 2n}{3} = \dfrac{n(n+2)}{3}$

38. (b): Since, $(r+1)^{\text{th}}$ term in the expansion of $(x+a)^n$ is $T_{r+1} = {}^nC_r (x)^{n-r} a^r$

$\therefore \; T_{11} = T_{10+1} = {}^{14}C_{10}(x)^{14-10}\left(\dfrac{1}{\sqrt{x}}\right)^{10}$

$= \left({}^{14}C_{10}\right)\dfrac{1}{x} = \dfrac{1001}{x}$

39. (c) : Since, $\vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow \vec{c} = -(\vec{a} + \vec{b})$

$\Rightarrow |\vec{c}|^2 = |-(\vec{a} + \vec{b})|^2$

$\Rightarrow |\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow 49 = 9 + 25 + 2(3)(5)\cos\theta$

$\Rightarrow \dfrac{15}{30} = \cos\theta \Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$

40. (c) : It is given that each vector $\vec{a}, \vec{b}$ and $\vec{c}$ is perpendicular to the sum of the remaining.

$\therefore \; \vec{a}\cdot(\vec{b} + \vec{c}) = 0, \vec{b}\cdot(\vec{a} + \vec{c}) = 0, \vec{c}\cdot(\vec{a} + \vec{b}) = 0$

$\Rightarrow \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = 0, \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{c} = 0, \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} = 0$

Adding all these values, we get

$2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$ ...(i)

Now, $|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$
$\qquad\qquad\qquad\qquad + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$

$= 9 + 16 + 25 + 0 = 50$

$\Rightarrow |\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$

41. (c) : We have, $2x + 3y - 3 = 0$ ...(i)

Slope of line (i), $m_1 = \dfrac{-2}{3}$

Also, $x + ky + 7 = 0$ ...(ii)

Slope of line (ii), $m_2 = \dfrac{-1}{k}$

Since the lines (i) and (ii) are perpendicular

$\therefore \; \left(\dfrac{-2}{3}\right) \times \dfrac{-1}{k} = -1 \Rightarrow k = -\dfrac{2}{3}$

42. (d): Area of circle, $A = \pi r^2$ ...(i)
Differentiate (i) w.r.t r, we get

$$\frac{dA}{dr} = 2\pi r$$

$$\therefore \left(\frac{dA}{dr}\right)_{r=2} = 2\pi(2) = 4\pi$$

43. (c): $\tan\dfrac{\pi}{8} = \dfrac{\sin\dfrac{\pi}{8}}{\cos\dfrac{\pi}{8}} \times \dfrac{\cos\dfrac{\pi}{8}}{\cos\dfrac{\pi}{8}}$

$$= \frac{2\sin\dfrac{\pi}{8}\cos\dfrac{\pi}{8}}{2\cos^2\left(\dfrac{\pi}{8}\right)} = \frac{\sin\left(\dfrac{\pi}{4}\right)}{\cos\left(\dfrac{\pi}{4}\right)+1}$$

$$= \frac{\dfrac{1}{\sqrt{2}}}{\dfrac{1}{\sqrt{2}}+1} = \frac{1}{\sqrt{2}+1}$$

44. (a): Given curves are $y^2 = 2x$ and $y = x$
Their points of intersection are $(0, 0)$ and $(2, 2)$
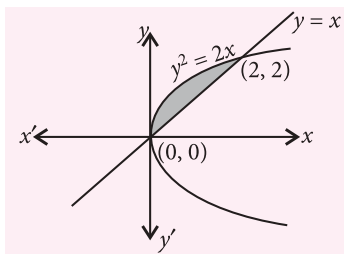
$\therefore$ Required area $= \displaystyle\int_0^2\left(y - \frac{y^2}{2}\right)dy = \left(\frac{y^2}{2} - \frac{y^3}{6}\right)\Bigg|_0^2$

$$= \left(2 - \frac{8}{6}\right) = \frac{4}{6} = \frac{2}{3} \text{ sq. units}$$

45. (c): $P(A \cap B) = \dfrac{7}{10}$, $P(B) = \dfrac{17}{20}$

$$\therefore\ P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{7}{10} \times \frac{20}{17} = \frac{14}{17}$$

46. (a, d): $CV_1 = 60$, $CV_2 = 70$, $\sigma_1 = 21$, $\sigma_2 = 16$

Let $\bar{x}_1$, $\bar{x}_2$ be the means of 1st and 2nd distribution respectively.

$$\Rightarrow CV_1 = \frac{\sigma_1}{\bar{x}_1} \times 100$$

$$\Rightarrow \bar{x}_1 = \frac{\sigma_1}{CV_1} \times 100 = \frac{21}{60} \times 100 = 35$$

Similarly, $\bar{x}_2 = \dfrac{\sigma_2}{CV_2} \times 100 = \dfrac{16}{70} \times 100 = 22.85$

47. (b): Required probability $= \dfrac{^4C_2}{^{52}C_2} = \dfrac{4 \times 3}{52 \times 51} = \dfrac{1}{221}$

48. (a): Given $\sin^{-1}x + \sin^{-1}y = \dfrac{\pi}{2}$
let $\sin^{-1}x = t \Rightarrow x = \sin t$

$$\Rightarrow t + \sin^{-1}y = \frac{\pi}{2} \Rightarrow \sin^{-1}y = \frac{\pi}{2} - t$$

$$\Rightarrow y = \sin\left(\frac{\pi}{2} - t\right) = \cos t$$

$$\therefore\ x^2 = \sin^2 t = 1 - \cos^2 t = 1 - y^2$$

49. (d): Let $I = \displaystyle\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}dx$ ...(i)

$$= \int_2^8 \frac{\sqrt{10-10+x}}{\sqrt{10-x} + \sqrt{10-10+x}}dx$$

$$I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x} + \sqrt{x}}dx$$ ...(ii)

Adding (i) & (ii), we get

$$2I = \int_2^8 dx = [x]_2^8 = 6 \Rightarrow I = 3$$

50. (d): Converse of the given statement is "If x is odd number then x is prime".
Contrapositive of the converse of the given statement statement is "If x is not a prime number then x is not an odd".

51. (c): Let A be the event of getting a total score of 5.
$\therefore\ A = \{(1, 4), (4, 1), (2, 3), (3, 1)\}$

Required probability $= \dfrac{4}{36} = \dfrac{1}{9}$

52. (a): Given, $A = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

$$\Rightarrow A^T = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

Also, $A + A^T = I$

$$\Rightarrow \begin{bmatrix} 2\cos 2\theta & 0 \\ 0 & 2\cos 2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On comparing, we get

$$2\cos 2\theta = 1 \Rightarrow \cos 2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

53. (d): Given, $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$\therefore A^2 - 5A = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$

$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = -7I$

54. (a): Since $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector

$\therefore |x\hat{i} + x\hat{j} + x\hat{k}|^2 = 1$

$\Rightarrow 3|x|^2 = 1 \Rightarrow x^2 = \dfrac{1}{3} \Rightarrow x = \pm\dfrac{1}{\sqrt{3}}$

55. (b): We have, $x = 2 + 3\cos\theta \Rightarrow 3\cos\theta = x - 2$ ...(i)
$y = 1 - 3\sin\theta \Rightarrow 3\sin\theta = -y + 1$ ...(ii)
Squaring adding (i) & (ii), we get
$(x - 2)^2 + (-(y - 1))^2 = 3^2 (\cos^2\theta + \sin^2\theta)$
$\Rightarrow (x - 2)^2 + (y - 1)^2 = 3^2$
$\therefore$ Radius = 3, centre = (2, 1)

56. (a): Since, $d = \dfrac{3}{\sqrt{14}}$ and $\vec{n} = 2\hat{i} - 3\hat{j} + \hat{k}$

$\therefore \hat{n} = \dfrac{2\hat{i} - 3\hat{j} + \hat{k}}{|2\hat{i} - 3\hat{j} + \hat{k}|} = \dfrac{1}{\sqrt{14}}(2\hat{i} - 3\hat{j} + \hat{k})$

$\therefore$ Required vector equation of plane is $\vec{r} \cdot \hat{n} = d$

$\Rightarrow \vec{r} \cdot \dfrac{(2\hat{i} - 3\hat{j} + \hat{k})}{\sqrt{14}} = \dfrac{3}{\sqrt{14}}$

$\Rightarrow \vec{r} \cdot (2\hat{i} - 3\hat{j} + \hat{k}) = 3$

57. (d): The equation of plane is $5y + 8 = 0$
$\therefore$ Equation of normal to the plane is
$\dfrac{5}{\sqrt{25}} y = \dfrac{-8}{\sqrt{25}} \Rightarrow -y = \dfrac{8}{5}$
$\therefore$ Direction cosines of normal drawn from origin to
the given plane are $(0, -1, 0)$ and $d = \dfrac{8}{5}$ units
$\Rightarrow$ Coordinates of foot of perpendicular drawn from
origin are $\left(0, \dfrac{-8}{5}, 0\right)$

58. (c): We have, $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$
$= 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1$
$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3$
$= 2 \times 1 - 3 = 2 - 3 = -1$
$\qquad [\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1]$

59. (a): $\sin 1° + \sin 2° + ... + \sin 359°$
$= \sin 1° + \sin 2° + ... + \sin 180° + ...$
$\qquad\qquad\qquad + \sin(360° - 2°) + \sin(360° - 1°)$
$= \sin 1° + \sin 2° + ... + \sin 180° + ... - \sin 2° - \sin 1°$
$= 0$

60. (c): Given differential equation is,
$x\dfrac{dy}{dx} - y = x^4 - 3x \Rightarrow \dfrac{dy}{dx} - \dfrac{y}{x} = x^3 - 3$
$\therefore$ Integrating factor $= e^{\int \frac{-1}{x}dx} = e^{-\log x} = e^{\log(x)^{-1}} = \dfrac{1}{x}$